An application of operator *-algebras to the theory of quantum measurement

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AN APPLICATION OF OPERATOR *-ALGEBRAS TO THE THEORY OF QUANTUM MEASUREMENT

A Thesis

presented for the Degree of Doctor of Philosophy

in Mathematics in the Open University

by

Juan Sotelo-Campos , B. Sc.

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ABSTRACT OF THESIS

In this thesis we adapt the Davies and Lewis' operational approach to quantum probability to the Op*-algebra framework.

In Chapter I we introduce our formalism of quantum mechanics. We define the space of pure states W as a nuclear-Frechet space generated by a self-adjoint operator M whose inverse is nuclear. The algebra of observables is taken to be $\mathcal{L}^{+}(W)$, the adjoint-stable set of linear maps from W into itself. This algebra and its dual under the uniform topology form a dual pair. We derive several algebraic and topological properties of this dual pair and show that it forms a suitable structure for the adaptation of Davies and Lewis' theory to the Op*-algebra framework.

In Chapter II we set the basis of the Davies and Lewis' operational approach to quantum probability within the framework developed in chapter I. Here we define the fundamental concepts of an expectation and an instrument and discuss their relationship. We show that instruments are bounded Radon measures and that the compose of two of them is not in general a stable operation on the universe of all instruments. We conclude the chapter discussing the general Robertson-Heisenberg uncertainty relations.

In Chapter III we build a special class of instruments to measure Q, P and H on $S(R)$; these are based on Davies' approximate position measurements. In particular we show that the compose of any two, hence any finite number, of these instruments is an instrument.

ORIGINALITY

In the rewriting of this thesis the author has closely follow the joint paper of his and D.A. Dubin [83].

It is worth pointing out at the outset that for Chapters II and III the Operational approach of Davies and Lewis and the theory of repeated measurements of Davies served as prototypes. Thus the originality claimed below is a fortiori within these boundaries.

In Chapter I originality is claimed for Definition I.6, Example I.14.(b), Propositions I.7(parts (a) and (b)), 8, 13, 23, 26(except eqn (60)), 29, 30 ; Lemmas I.17, 24, 27, and Corollary I.30.

Definition I.10; Propositions I.11, 23; and Corollary I.12 are due to Dr. D.A. Dubin.

In Chapter II the setting of the operational approach to quantum probability of Davies and Lewis within the Op*-algebra framework even though quite straightforward is new. It is worth mentioning here that the name $(\mathcal{A},W)$, $\mathcal{A}$-measure, and the notions of pre and post-instruments are due to Dr. D.A. Dubin.

In Chapter III originality is claimed for Definition III.5, Lemmas III. 9(part (v)), 11(its last part), 12, 14, 15 ,19 ,20; Proposition III.1,18;and Corollaries III. 2, 10, 13, 16.

The first part of Lemma III.11 is due to Dr. D.A. Dubin.

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CHAPTER I

FORMALISM OF QUANTUM MECHANICS

This chapter is devoted to the establishing of a natural setting for the extension of the Davies & Lewis' theory of quantum measurement [1,2] to the case of families of unbounded observables. We shall be making an extensive use of the theory of topological vector spaces (tvs), particularly, of the duality theory. The content of this chapter forms a natural background for all of what follows in this work. We proceed to survey the chapter briefly.

We begin the chapter with a short review of the basic notions of Op*-algebras of (in general) unbounded operators (I.1-5). We then go on to describe our formalism of quantum mechanics.To that end in (I.6) we introduce the definition of a state space and then proceed to investigate its topological properties (I.7-9). In (I.10-12) we discuss a class of state spaces which are particularly frequent in applications, namely spaces of type S^d. (I.16-20) present the uniform topology <u> on the algebra of observables $\mathcal{A}$,and study the algebraic and order structure of $\mathcal{A}$ as well as the relation existing between <u>and this order structure . Sections (I.21-23) are concerned with the dual of $\mathcal{A}[u]$, $\mathcal{A}'$, and its algebraic and order structure. There the elements of $\mathcal{A}'$ are characterized as a particular type of trace class operators. Sections (I.24-26) presents the dual pair $\langle \mathcal{A}, \mathcal{A}' \rangle$ and discuss the several topologies on $\mathcal{A}$ and $\mathcal{A}'$ originated by this structure. Finally, (I.26-30) give a somewhat detailed account of the properties of $\mathcal{A}'$ furnished with the strong topology β^*.

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A REVIEW OF OP*-ALGEBRAS

*-algebras of unbounded operators in Hilbert spaces play an important role in quantum physics, in particular in quantum field theory. The fields describing quantum systems with infinitely many degrees of freedom are families of unbounded operator-valued distributions on various representation Hilbert spaces ([57]). For relativistic systems a model of the fields as representations of a topological *-algebra has been proposed by Borchers ([58]) and Uhlmann ([59]) and considered by various authors since; ([60-61]) are review articles. In [68] Dubin and Alcantara have considered similar models for nonrelativistic systems and current algebras. To that effect they introduced the so called I*-algebras: complex unital topological *-algebras which are nuclear LF-spaces and for which $\sum_J x_j^* x_j = 0 \Rightarrow x_j = 0$ $(j \in J)$, J any finite subset of $\mathbb{N}$. It was shown in [69] that an important type of I*-algebras, namely $\underline{E}_\otimes = \sum_{n \geq 0}^{\oplus} \otimes^n E[\tau]$; the tensor algebra over a nuclear Frechet space $E[\tau]$, could be realized as an Op*-algebra on a domain $\mathcal{D} = \mathcal{B}_\infty(S) = \bigcap_{n \in \mathbb{N}_0} D(S^n)$, S a certain self-adjoint operator in a suitable HIlbert space. *-algebras of unbounded operators in Hilbert spaces have also a natural application to quantum systems of a finite degree of freedom. We recall that one of the special circumstances on which the Von Neumann scheme for quantum mechanics was based is that the pure states of the system constitute a Hilbert space ([65]). An operator algebraic scheme for quantum theory which goes beyond

this hypothesis has been available for a few years now. We have in mind here unbounded operator algebras. As early work in this direction we single out the papers of Roberts ([45-46]) and Kristensen, Mejlbo and Thue Poulsen ([70]). Here we find a theory in which the pure states constitute not a Hilbert space, but rather a topological vector space continuously and densely embedded in a Hilbert space, a rigged triple in the sense of Gelfand ([71]). Subsequently, the theory of operator algebras associated with such spaces has received attention. Its present form was developed in the scholarly work of Lassner et al. ([4-9], [13,14,18]) and Schmüdgen ([10-12]), although some other investigators have made important contributions ([15-17, 19]).

I.1 <u>DEFINITION:</u> Let $\mathfrak{D}$ be a dense linear subspace of a separable Hilbert space $\mathcal{H}$. By $\mathcal{L}^+(\mathfrak{D})$ we denote the set of all linear operators (a) with domain D(a) which satisfy the following conditions.

(a) $\mathfrak{D} \subseteq D(a)$, $a\mathfrak{D} \subseteq \mathfrak{D}$

(b) The adjoint operator a* exists and

$\mathfrak{D} \subseteq D(a^*)$, $a^*\mathfrak{D} \subseteq \mathfrak{D}$

These operators are not necessarily bounded.

For any two elements (a,b) in $\mathcal{L}^+(\mathfrak{D})$ the applications

(c) $\mathcal{L}^+(\mathfrak{D}) \times \mathcal{L}^+(\mathfrak{D}) \longrightarrow \mathcal{L}^+(\mathfrak{D})$ (1)

$(a,b) \longmapsto ab$ defined by

$(ab)\varphi = a(b\varphi)$ for all φ in $\mathfrak{D}$,

and

(d) $\mathcal{L}^+(\mathcal{D}) \longrightarrow \mathcal{L}^+(\mathcal{D})$

$a \longmapsto a^+$ defined by

$a^+\varphi = a^*\varphi$ for all φ in $\mathcal{D}$, i.e. $a^+ = a^*\restriction\mathcal{D}$,

define, respectively, a multiplication and an involution on $\mathcal{L}^+(\mathcal{D})$. ///

With the multipication and involution just defined $\mathcal{L}^+(\mathcal{D})$ becomes an $*$-algebra.

For $\mathcal{D} = \mathcal{H}$ we have by the closed graph theorem $\mathcal{L}^+(\mathcal{D}) = \mathbb{B}(\mathcal{H})$, the $*$-algebra of all bounded operators on $\mathcal{H}$.

I.2 <u>DEFINITIONS:</u>

(a) A $*$-subalgebra $\mathcal{A}$ of $\mathcal{L}^+(\mathcal{D})$ containing the identity I will be called an <u>Op*-algebra</u>.

Set $\underline{\mathcal{D}}(\mathcal{A}) = \bigcap_{a\in\mathcal{A}} D(\bar{a})$ and $\mathcal{D}_*(\mathcal{A}) = \bigcap_{a\in\mathcal{A}} D(a^*)$,

where $\bar{a}$ is the closure of the linear operator (a) in the Hilbert space $\mathcal{H}$ and $\mathcal{A}$ is an Op*-algebra.

(b) $\mathcal{A}$ is said to be closed on $\mathcal{D}$ if $\mathcal{D} = \underline{\mathcal{D}}(\mathcal{A})$.

(c) The domain $\mathcal{D}$ is said to be closed if $\mathcal{L}^+(\mathcal{D})$ is closed on $\mathcal{D}$.

(d) $\mathcal{A}$ is said to be self-adjoint on $\mathcal{D}$ if $\mathcal{D} = \mathcal{D}_*(\mathcal{A})$. ///

After these preliminary definitions we now pass on to discuss the topologization of the domain $\mathcal{D}$. A suitable choice of a topology on $\mathcal{D}$ will allow us to give a simple characterization of several topologies on Op*-algebras defined on $\mathcal{D}$.

I.3 <u>DEFINITION:</u> Let $\mathcal{A}$ be an Op*-algebra on $\mathcal{D}$, by $t_{\mathcal{A}}$ we denote the weakest locally convex topology on $\mathcal{D}$ with respect to which every operator $a\in\mathcal{A}$ is a continuous linear mapping of the locally convex space $\mathcal{D}[t_{\mathcal{A}}]$ into $\mathcal{D}[t_{\mathcal{H}}]$, where $t_{\mathcal{H}}$ is the Hilbert space topology on $\mathcal{D}$ inherited from $\mathcal{H}$.

If $\mathcal{A}=\mathcal{L}^{+}(\mathcal{D})$ then we write t_{+} instead of $t_{\mathcal{L}^{+}(\mathcal{D})}$. ///

As an immediate consequence of this definition we have

I.4 <u>PROPOSITION</u> <u>([4]):</u>

(a) The topology $t_{\mathcal{A}}$ is given by the seminorms, for any $\varphi\in\mathcal{D}$

$$\|\varphi\|_a=\|a\varphi\| \quad , \quad a\in\mathcal{A} \tag{2}$$

and is stronger than the Hilbert space topology $t_{\mathcal{H}}$, since by definition $\mathcal{A}$ contains the identity operator (I.2.a).

(b) Every operator $a\in\mathcal{A}$ is a continuous linear map of the locally convex space $\mathcal{D}[t_{\mathcal{A}}]$ into itself.

(c) If every operator $a\in\mathcal{A}$ is bounded, then $t_{\mathcal{A}}$ coincides with the Hilbert space topology $t_{\mathcal{H}}$ defined by the norm $\|.\|$. ///

The topology $t_{\mathcal{A}}$ is known as the <u>graph</u> <u>topology</u>. This topology can also be generated by the family of seminorms $\{p_a : a\in\mathcal{A}\}$, where p_a is defined by, for any $\varphi\in\mathcal{D}$,

$$p_a(\varphi) = \|a\varphi\| + \|\varphi\| \tag{3}$$

This can be seen as follows: For any $a\in\mathcal{A}$ $\|\varphi\|_a\leqslant p_a(\varphi)$ for all $\varphi\in\mathcal{D}$. On the other hand, given an $a\in\mathcal{A}$ we find $c=(1/\sqrt{2})(a^{+}a + I) + I$ belonging to $\mathcal{A}_h$ so that, for all $\varphi\in\mathcal{D}$, $\|\varphi\|_c\geqslant p_a(\varphi)$.

The locally convex space $\mathcal{D}[t_{\mathcal{A}}]$ is in general not complete. Its completion is denoted by $\tilde{\mathcal{D}}[\tilde{t}_{\mathcal{A}}]$ and satifies ([4:3.2])

$$\tilde{\mathcal{D}}[\tilde{t}_{\mathcal{A}}] = \bigcap_{a\in\mathcal{A}} D(\bar{a}) = \underline{\mathcal{D}}(\mathcal{A}) \qquad (4)$$

Therefore an Op*-algebra $\mathcal{A}$ is closed iff $\mathcal{D}[t_{\mathcal{A}}]$ is complete.

Now from definition I.3 we see that $\mathcal{D}[t_{\mathcal{A}}]$ is continuously embedded into $\mathcal{H}$. Let us denote by $\mathcal{D}'[t^{\mathcal{A}}]$ the strong dual of $\mathcal{D}[t_{\mathcal{A}}]$. If $\mathcal{A} = \mathcal{L}^+(\mathcal{D})$ we shall write t_+' instead of $t^{\mathcal{L}^+(\mathcal{D})}$. Writing the linear functionals $F\in\mathcal{D}'$ on $\mathcal{D}$ in the form $\langle F,\varphi\rangle$, we equip $\mathcal{D}'$ with the anti-linear structure defined by $\langle\lambda F + \mu G,\varphi\rangle = \bar{\lambda}\langle F,\varphi\rangle + \bar{\mu}\langle G,\varphi\rangle$. Then $\psi \longmapsto F_\psi$ defined by $\langle F_\psi,\varphi\rangle = \overline{\langle\psi,\varphi\rangle}$ for $\psi\in\mathcal{H}$, defines a linear embedding of $\mathcal{H}$ into $\mathcal{D}'$. Therefore we obtain, in a natural way, a <u>rigged Hilbert space</u> structure

$$\mathcal{D}[t_{\mathcal{A}}] \subset \mathcal{H} \subset \mathcal{D}'[t^{\mathcal{A}}]\ . \qquad (5)$$

Consequently, we have on $\mathcal{D}$ three topologies

$$t_{\mathcal{A}} \geqslant \|\cdot\| \geqslant t^{\mathcal{A}} \qquad (6)$$

in correspondence with the rigged Hilbert space defined by $\mathcal{A}$. Some properties of $\mathcal{D}[t^{\mathcal{A}}]$ and $\mathcal{D}'[t^{\mathcal{A}}]$ are collected in the following proposition.

I.5 <u>PROPOSITION ([13])</u>: Let $\mathcal{D}$ be a closed domain and $\mathcal{A}$ be a closed Op*-algebra on $\mathcal{D}$. Then

(a) $\mathcal{D}[t_{\mathcal{A}}]$ is semirreflexive.

(b) The topologies $t_{\mathcal{A}}$ and t_+ have the same bounded sets, hence

$$(\mathcal{D}[t_{\mathcal{A}}])'[t_+' \restriction \mathcal{D}[t_{\mathcal{A}}]'] = \mathcal{D}'[t^{\mathcal{A}}]$$

(c) $\mathfrak{D}'[t^A]$ is barrelled and bornological.

Furthermore the following assertions are equivalent

(d) $\mathfrak{D}'[t^A]$ is reflexive.

(e) $\mathfrak{D}[t_A]$ is barreled.

(f) t_A is the Mackey topology and $\mathfrak{D}'[t^A]$ is reflexive. ///

We have now introduced all the basic concepts necessary for the characterization of our Quantum mechanical systems' structure.

FORMALISM OF QUANTUM MECHANICS

The class of quantum mechanical systems we shall be considering is determined by the following definition of pure states. It will be seen to include all the usual non-relativistic systems of particles.

I.6 DEFINITION: The space $W[t]$ of wave functions for a system with d degrees of freedom is a locally convex space such that

(a) There exists a self-adjoint operator M on a separable Hilbert space $\mathcal{H}$ so that the embedding map $\Psi_1 : (D(M), \|\cdot\|_1) \longrightarrow \mathcal{H}$ is nuclear and

$$W = \bigcap_{n \in \mathbb{N}_0} D(M^n) \quad . \tag{7}$$

Here the norm $\|\cdot\|_1$ on the Hilbert space $(D(M), \|\cdot\|_1)$ is given by $\|\varphi\|_1 = [\|M\Psi_1\varphi\|^2 + \|\Psi_1\varphi\|^2]^{1/2}$.

(b) The topology t is given by the projective topology with respect to the mappings

$$\Psi_n : (D(M^n), \|\cdot\|_n) \rightarrow \mathcal{H} \qquad , n \in \mathbb{N}_0 \ , \tag{8}$$

where the norm $\|\cdot\|_n$ on the Hilbert space $(D(M^n), \|\cdot\|_n)$ is obtained from the inner product

$$\langle\varphi,\psi\rangle_n = \sum_{i=0}^{n} \langle M^i\Psi_n\varphi, M^i\Psi_n\varphi\rangle \qquad (9)$$

so that

$$\|\varphi\|_n = [\sum_{i=0}^{n}\|M^i\Psi_n\varphi\|^2]^{1/2} \qquad (10)$$

and $M^0 = I$ by definition.

In other words

$$W[t] = \underleftarrow{\text{lim proj}}\ (D(M^n), \|\cdot\|_n) \qquad (11)$$

and t is generated by the seminorms $\|\cdot\|_n$, $n\in\mathbb{N}_0$. ///

This type of spaces was analyzed by Pietch [35] and Lassner & Timmermann [8], who called W[t] a space of type $\mathcal{D}_\infty$.

The next proposition gives a more or less complete characterization of W[t] and its strong dual W'[t'].

I.7 <u>PROPOSITION:</u>

(a) W[t] is nuclear and Frechet. Hence it is barreled, bornological, Mackey, Montel, reflexive and separable.

(b) W'[t'] is nuclear DF and complete, barreled, bornological, Mackey, Montel, reflexive and separable.

(c) The self-adjoint operator M possesses a complete countably orthonomal system of eigenvectors $\{e_n : n\in\mathbb{N}\}$ which form an absolute basis of W[t]. Consequently W[t] is tvs-isomorphic to the Köthe space

$$\ell_2(\langle e_n,e_n\rangle_m) = \{(d_n): \sum_{n\geq 1}|d_n|^2\langle e_n,e_n\rangle_m = \|(d_n)\|_m^2 < \infty, m\in\mathbb{N}_0\} \quad (12)$$

(d) In Gelfand's sense

$$W[t] \hookrightarrow \mathcal{H} \hookrightarrow W'[t'] \qquad (13)$$

constitutes a rigged triple.

PROOF.

(a,b) Set $D_n = (D(M^n), \|.\|_n)$, from the definition of the maps Ψ_n we see that for $n<m$

$$\Psi_m = \Psi_n \circ \Psi_{m,n} \qquad (14)$$

where $\Psi_{m,n}$ is the canonical inclusion of D_m into D_n. To show the nuclearity of $W[t]$ it will be sufficient to show that $\Psi_{m,n}$ is a nuclear map whenever $n<m$. ([20:III.7.3(cor 3)]). First we show that Ψ_m, $m\in\mathbb{N}$ is nuclear.

For any $\varphi\in D_m$ and $m>n$

$$\|\Psi_{m,n}\varphi\|_n^2 = \sum_{i=0}^{n}\|M^i\Psi_n(\Psi_{m,n}\varphi)\|^2 = \sum_{i=0}^{n}\|M^i\Psi_m\varphi\|^2 \leq \sum_{i=0}^{m}\|M^i\Psi_m\varphi\|^2 = \|\varphi\|_m^2 \qquad (15)$$

hence $\Psi_{m,n}$ is continuous. Since by (14) $\Psi_m = \Psi_1\circ\Psi_{m,1}$ $m>1$, and by hypothesis Ψ_1 is nuclear, then Ψ_m, $m\in\mathbb{N}$, is nuclear ([20:III.7.1(cor 2)]).

Now, for $m>n$, and any $\varphi\in D_m$, $\psi\in D_n$

$$\begin{aligned}
\langle\Psi_{m,n}\varphi,\psi\rangle_n &= \sum_{i=0}^{n}\langle M^i\Psi_n(\Psi_{m,n}\varphi), M^i\Psi_n\psi\rangle \\
&= \sum_{i=0}^{n}\langle M^{2i}\Psi_m\varphi, \Psi_n\psi\rangle \\
&= \langle(\sum_{i=0}^{n} M^{2i})\Psi_m\varphi, \Psi_n\psi\rangle \\
&= \langle\Psi_n^\times(\sum_{i=0}^{n} M^{2i})\Psi_m\varphi, \psi\rangle_n
\end{aligned}$$

where $\Psi_n^\times$ is the adjoint of Ψ_n. Recall that as $\Psi_n : D_n \longrightarrow H$ then $\Psi_n^\times : H \longrightarrow D_n$, and for any $\xi\in H$ and $\theta\in D_n$ $\langle\Psi_n^\times\xi,\theta\rangle_n = \langle\xi,\Psi_n\theta\rangle$.

Therefore

$$\Psi_{m,n} = \Psi_n^\times (\sum_{i=0}^{n} M^{2i})\Psi_m. \qquad (16)$$

Since Ψ_m is continuous (by (10), $\|\Psi_m\varphi\|^2 \leq \|\varphi\|_m$, for any m) and $\sum_{i=0}^{n} M^{2i}$ is closed on $\Psi_n(D_n)\supset\Psi_m(D_m)$ (it is self-adjoint), then $(\sum_{i=0}^{n} M^{2i})\Psi_m : D_m \longrightarrow H$ is continuous.

On the other hand Ψ_n is nuclear if and only if $\Psi_n^{\times}$ is nuclear ([73:X.3]). It now follows from (16) and [20:III.7.1 (cor 2)] that $\Psi_{m,n}$ is nuclear whenever $n<m$, so we are done.

The Frechet property follows from the fact that W[t] is a projective limit of a countable collection of Hilbert spaces [20:II.5.3].

Since W[t] is nuclear and Frechet the list of its topological properties and that of its strong dua l W'[t'] follow from standard results in tvs theory, cf[40:33.2,36.3,56.14],[20:II.9.1,IV.5.7,IV.6.6] and [72:4.3.3,4.4.10,4.4.12].

(c) For a proof see [35:4.3]

(d) This follows directly from the definition of W[t] and the fact that $\mathcal{H}$ is tvs-isomorphic to its strong dual $\mathcal{H}'$. ∎

A few remarks concerning the self-adjoint operator M are to the point. As was established in I.7.c M possesses a complete orthonormal system of eigenvectors $\{e_r : r\in\mathbb{N}\}$.

For each one of these eigenvectors ve have

$$Me_r = \lambda_r e_r \quad , \tag{17}$$

λ_r being the corresponding eigenvalue.

These λ_r satisfy the folowing relations ([35])

$$\lim_{r\to\infty} |\lambda_r| = +\infty \quad , \tag{18.a}$$

and

$$\sum_{r=1}^{\infty} \frac{1}{1+|\lambda_r|^{\rho}} < +\infty \tag{18.b}$$

for some suitable $\rho \in \mathbb{N}$.

The former follows from the fact that in $W[t]$ every bounded set is precompact ([20:III.7(cor 2)]) while the latter follows from the nuclearity of the applications $\Psi_{m,n}$ $(m>n)$.

As the next proposition shows we may asssume, without loss of generality, that $M \geqslant I$ and possesses an inverse M^{-1} that is nuclear.

I.8 <u>PROPOSITION:</u> Let M be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$, then the embedding map $\Psi_1 : (D(M), \| \ \|_1) \longrightarrow \mathcal{H}$ is nuclear if and only if the inverse of the self-adjoint operator $I + M^2$ is nuclear.

<u>PROOF.</u>

For any $\xi \in \mathcal{H}$, $\theta \in \Psi_{21}(D_2)$

$$\langle \xi , \Psi_1 \theta \rangle = \langle \Psi_1^{\times} \xi , \theta \rangle_1 = \sum_{i=0}^{1} \langle M^i \Psi_1 (\Psi_1^{\times} \xi), M^i \Psi_1 \theta \rangle$$
$$= \langle (\Psi_1 \Psi_1^{\times}) \xi , (I+M^2) \Psi_1 \theta \rangle .$$

Now $\Psi_1[\Psi_{21}[D_2]] = (\Psi_1 \circ \Psi_{2,1})[D_2] = \Psi_2[D_2] = D(M^2) = D(I+M^2)$. Hence, $\Psi_1 \Psi_1^{\times} \xi \in D((I+M^2)^{*}) = D(I+M^2)$ and

$$(I + M^2) \Psi_1 \Psi_1^{\times} \xi = \xi$$

Therefore $\Psi_1 \Psi_1^{\times} = (I + M^2)^{-1} : \mathcal{H} \longrightarrow \mathcal{H}$ and so $(\Psi_1 \Psi_1^{\times})^{1/2} = (I + M^2)^{-1/2}$. Now, $(\Psi_1 \Psi_1^{\times})^{1/2}$ is nuclear if and only if Ψ_1 is nuclear ([73:X.3]), hence $(I+M^2)^{-1/2}$ is nuclear if and only if Ψ_1 is nuclear. But $(I+M^2)^{-1/2}$ is nuclear if and only if $(I+M^2)^{-1}$ is nuclear ([73:X.3]), and we are done. ∎

I.9 <u>NOTATION:</u> From now on we shall abbreviate $\mathcal{L}^+(W)$ to $\mathcal{A}$, and assume that the operator M defining $W[t]$ satisfies

$$M \geqslant I \quad \text{and} \quad M^{-1} \text{ is nuclear.}$$

The topology on W we have considered so far, is the natural one obtained through the very same operator (and its positive integer powers) that generated the domain. Let us now consider on W the graph topology t_+ (2) generated by the Op*-algebra $\mathcal{A}$. It turns out, as the next proposition shows, that t_+ is equivalent to t, i.e. the graph topology is already given by the system of seminorms $\{\|\cdot\|_n : n \in \mathbb{N}_0\}$ defined by (10) (c.f. also [47])

<u>PROPOSITION:</u> The graph topology t_+ is equivalent to the topology t.

<u>PROOF.</u>

We only prove the second part as $M^n \in \mathcal{A}$ for any $n \in \mathbb{N}_0$, immediately implies $t \leqslant t_+$. Every $b \in \mathcal{A}$ is closable in $\mathcal{H}$ and hence closed as a map from $W[t]$ into $\mathcal{H}$. Thus $\langle b \rangle$ is a closed operator between Frechet spaces, hence continuous ([20:III.2.]). That is there is a constant $c>0$ and an index $n \in \mathbb{N}_0$ such that for all $\varphi \in W$, $\|b\varphi\| \leqslant c\,\|\varphi\|_n$. ∎

A very important class of wave function spaces is given in the following definition.

I.10 <u>DEFINITION:</u> A space $W[t]$ of wave functions for a system with d degrees of freedom is of class $\mathcal{K}$ if it is the maximal separable locally convex space such that

(a) There exists a t-continuous scalar product, $\langle\ ,\ \rangle$ on W. The completion of W with respect to this scalar product is a separable Hilbert space $\mathcal{H}$.

(b) There exist d pairs $(b_j, b_j^*)_{1\leq j\leq d}$ of linear operators mapping W into itself, adjoint with respect to the scalar product

$$\langle b_j\varphi,\psi\rangle = \langle\varphi, b_j^*\psi\rangle \qquad (\forall\varphi,\psi\in W;\ 1\leqslant j\leqslant d) \ , \qquad (19)$$

and satisfying the canonical commutation relations (CCR)

$$[b_j\ , b_k^*] = \delta_{jk} \qquad (1\leqslant j,k\leqslant d) \qquad (20)$$

strongly on W, other commutators vanishing.

(c) The topology t is the graph topology generated by $\mathcal{A}$.

The space is said to be irreducible if, in addition,

(d) There is a vector $\Omega_0\in W$, normalized by $\|\Omega_0\| = 1$, satisfying the Fock-Cook condition

$$b_j\Omega_0 = 0 \qquad (1\leqslant j\leqslant d) \qquad (21)$$

(e) $\mathcal{P}\Omega_0$ is dense in W, where $\mathcal{P}\subset\mathcal{A}$ is the algebra of all polynomials in the canonical operators $(b_j, b_j^*)_{1\leq j\leq d}$. Hence Ω_0 is cyclic for $\mathcal{P}$. ///

This choice of system was analyzed by Kristensen, Mejlbo, and Thue Poulsen [70], who called irreducible wave function spaces of class $\mathcal{K}$ spaces of type S^d, for reasons which will be immediately apparent.

I.11 PROPOSITION:

(a) Every wave function space of class $\mathcal{K}$ may be decomposed into a countable locally convex direct sum of irreducible spaces:

$$W = \sum_{n\geq 1}^{\oplus} W_n \tag{22}$$

(b) Any irreducible wave function space W of class $\mathcal{K}$ is tvs-isomorphic to Schwartz's space $S^d = S(\mathbb{R}^d)$ with its usual Frechet topology.

Defining

$$M = \sum_{j=1}^{d} b_j b_j^* \ , \tag{23}$$

$$W = \bigcap_{p\in\mathbb{N}_0} D(M^p) \tag{24}$$

The topology t is determined by the seminorms

$$\varphi \longmapsto \|\varphi\|_p = \| M^p \varphi \| \qquad (p\geq 0) \tag{25}$$

PROOF.

(a) The decomposition is effect by choosing an orthonormal basis $\{\Omega_n : n\geq 1\}$ for the subspace $\{\varphi : M\varphi = d\varphi\}$. Each Ω_n then serves as a cyclic vector for $\mathcal{P}$, thereby determining W_n [77:§4.4].

(b) The isomorphism of an irreducible W with S^d is shown in [77:§4.10;70].

The identification of S^d with $\bigcap_{p\in\mathbb{N}_0} D(M^p)$ is due to Simon [51], and is known as the N-representation.

The topological result, that the usual topology on S^d is equivalent to the topology t, is a result of the closed graph theorem applied to $\mathcal{A}$, in view of the fact that t is the coarsest locally convex topology with respect to which every $a\in\mathcal{A}$ is continuous. ∎

I.12 <u>COROLLARY:</u> An irreducible wave function space, W[t], of class $\mathcal{K}$ is a domain of type $\mathcal{D}_\infty$.

W possesses an unconditional basis, The Hermite functions: for all $\nu \in \mathbb{N}^d$, $\nu = (\nu_1, \nu_2, \ldots, \nu_d)$, these are defined to be

$$\Omega_\nu = \Pi_{1\le j\le d} (\nu_j!)^{-1/2} [b_j^*]^{\nu_j} \Omega_0 ; \tag{26}$$

they satisfy the well-known recurrence relations

$$\begin{aligned} b_j^* \Omega_j &= (\nu_j + 1)^{1/2} \Omega_{(\nu_1, \ldots, \nu_j+1, \ldots, \nu_d)} \\ b_j \Omega_j &= (\nu_j)^{1/2} [1 - \delta_{0,\nu_j}] \Omega_{(\nu_1, \ldots, \nu_j - 1, \ldots \nu_d)} \end{aligned} \tag{27}$$

<u>PROOF.</u>

The first assertion is clear from (23 - 25).

The Hermite functions are used to construct the tvs isomorphism between W and the sequence space s^d: if $\varphi = \sum c_\nu \Omega_\nu$ is an element of $\mathcal{H}$, then $\varphi \in W$ if and only if $(c_\nu) \in s^d$ [26,51,70,77,ibid]. ∎

In I.9 we noted that for any wave function space W[t] we shall write

$$\mathcal{A} = \mathcal{L}^+(W). \tag{28}$$

For terminological purposes only, we shall refer to $\mathcal{A}$ as our algebra of observables, and all elements of $\mathcal{A}$ as observables. There is no implication of physical measurability implied. Those elements of $\mathcal{A}$ which can be measured in our scheme, we shall refer to as physical observables.

Powers [58] has introduced a notion of self-adjointness for algebras of unbounded operators (see DEFINITION I.2), and $\mathcal{A}$ satisfies the requirements.

I.13 PROPOSITION: $\mathcal{A}$ is a complex unital $*$-algebra which is closed and self-adjoint, i.e.

$$W = \bigcap_{\mathcal{A}} D(\bar{a}) = \bigcap_{\mathcal{A}} D(a^*) \qquad (29)$$

PROOF.

The first condition follows from (4) since $W[t]$ is complete. The second condition follows because by I.7.c $W[t]$ is isomorphic to a Köthe sequence space ([8]). ∎

We shall now give some concrete examples of wave function spaces

I.14 EXAMPLES

(a) Let T^s be the s-dimensional torus defined by

$$0 \leq x_j \leq 2\pi \qquad j=1,\dots,s \quad , \; s \in \mathbb{N} \quad ,$$

and $L^2(T^s)$ be the space of square Lebesque-integrable functions on T^s. We take as W the space $\mathcal{D}[T^s]$ of C^∞-functions on T^s, i.e. φ is $C^\infty(\mathbb{R}^s)$ and its derivatives of all orders are periodic with period 2π in each of the variables x_j.

On $\mathcal{D}[T^s]$ define the differential operators

$$l_j\varphi = i\frac{\partial\varphi}{\partial x_j} \qquad j=1,\dots,s \qquad (30)$$

Each one of these operators is essentially self-adjoint on $\mathcal{D}[T^s]$. Furthermore $1-l^2 = 1-\sum_{j=1}^{s} l_j^2$ is essentially self-adjoint on $\mathcal{D}[T^s]$ and possesses an inverse $(1-\bar{l}^2)^{-1}$ that is nuclear in $L^2(T^s)$ ([15], [74]).

Topologize $\mathcal{D}[T^s]$ with the seminorms

$$\|\varphi\|_n = \|(1-L^2)^n\varphi\| \qquad n\in\mathbb{N}_0 , \tag{31}$$

then $\mathcal{D}[T^s]$ is a nuclear-Frechet space of type $\mathcal{D}_\infty$, c.f.[20:III.8]

(b) Consider the Hilbert space $L^2[a,b]$. Let D denote the set of functions on (a,b) so that φ' is absolutely continuous on (a,b), $\varphi''\in L^2[a,b]$, and

$$\varphi(a) = \lim_{x\to a^+}\varphi(x) \quad , \quad \varphi(b) = \lim_{x\to b^-}\varphi(x) \tag{32}$$

On D define the differential operator

$$\ell\varphi = -\frac{d}{dx}\left[(x-a)(b-x)\frac{d}{dx}\varphi\right] \tag{33}$$

ℓ is a self-adjoint operator in $L^2[a,b]$ ([73:V.6],[76]) and generates the space $\mathcal{D}([a,b]) = \bigcap_{n\in\mathbb{N}_0} D(\ell^n)$ of functions that are infinitely differentiable on the interval [a,b]. Moreover, on this space the system of seminorms

$$\|\varphi\|_n = [\sum_{i=0}^{n}\|\ell^i\varphi\|^2]^{1/2} , \; n\in\mathbb{N}_0 \tag{34}$$

defines a topology that is Frechet and nuclear ([76]).

(c) Consider the Hilbert space $L^2(\mathbb{R}^\ell)$. On the Schwartz space $S(\mathbb{R}^\ell)$ define the differential operator

$$H\varphi = (1+|\underline{x}|^2-\underline{\Delta})\varphi$$

This operator is essentially self-adjoint. Furthermore

$$S(\mathbb{R}^\ell) = \bigcap_{n\in\mathbb{N}_0} D(H^n)$$

and the seminorms

$$\|\varphi\|_n = [\sum_{k=0}^{n}\|H^k\varphi\|^2]^{1/2}$$

generates the usual Schwartz topology which is nuclear and Frechet (see chapter III for further discussion).

I.15 We are now going to investigate how the Op*-algebra $\mathcal{A}$ is related to the linear spaces $\mathcal{L}(W[t],W'[t'])$, $\mathcal{L}(W[t])$ and $\mathcal{L}(W'[t'])$ of continuous functions from $W[t]$ into $W'[t']$, $W[t]$ into itself and $W'[t']$ into itself respectively. A close analysis of how each one of these spaces relates to each other will allow us to define useful topologies on the algebra of observables and states.

We first notice that by I.4.b $\mathcal{A}$ is a subspace of $\mathcal{L}(W[t])$. Because of (13) the latter linear space can be considered as a subspace of $\mathcal{L}(W[t],W'[t'])$. Now for any $a\in\mathcal{L}(W[t],W'[t'])$ the adjoint operator $a^{+}\in\mathcal{L}(W[t],W'[t'])$ (recall that $W[t]$ is reflexive) is uniquely defined by

$$\langle a^{+}\varphi,\psi\rangle = \overline{\langle a\varphi,\psi\rangle} \qquad (35)$$

so that the map $a\longmapsto a^{+}$ is an involution in $\mathcal{L}(W[t],W'[t'])$. Because of the reflexivity of $W[t]$ we have $\mathcal{L}(W[t])^{+} = \mathcal{L}(W'[t'])$ which by (13) can be considered as a subspace of $\mathcal{L}(W[t],W'[t'])$. Hence

$$\mathcal{A} = \mathcal{L}(W[t])\cap\mathcal{L}(W'[t']) \qquad (36)$$

and we have the following chain of operator spaces

$$\begin{array}{ccccc} & \subset & \mathcal{L}(W[t]) & \subset & \\ \mathcal{A} & & & & \mathcal{L}(W[t],W'[t']) \\ & \subset & \mathcal{L}(W'[t']) & \subset & \end{array} \qquad (37)$$

$\mathcal{L}(W[t],W'[t'])$ is only a linear space of operators with an involution, $\mathcal{L}(W[t])$ and $\mathcal{L}(W'[t'])$ are algebras of operators but not *-algebras, and $\mathcal{A}$ is a *-algebra.

If $W[t]=\mathcal{H}$, then all four spaces of linear operators coincide with the C*-algebra $B(\mathcal{H})$ of all bounded operators in $\mathcal{H}$, since in that case $W[t]=\mathcal{H}=W'[t']$. Otherwise, all four spaces (37) are muttually different. Let us further remark that $\mathcal{A}$ is an extremal $*$-algebra in $\mathcal{L}(W[t],W'[t'])$, i.e. there is no larger $*$-subalgebra of $\mathcal{L}(W[t],W'[t'])$ containing $\mathcal{A}$. But $\mathcal{A}$ is not maximal in the sense that it contains any other$*$-subalgebra of $\mathcal{L}(W[t],W'[t'])$, for instance the $*$-algebra $\mathcal{B}(\mathcal{H})$ is contained in $\mathcal{L}(W[t],W'[t'])$ and different from $\mathcal{A}$ if $W[t]\neq\mathcal{H}$.

TOPOLOGIES ON THE ALGEBRAS OF OBSERVABLES AND STATES

In what follows we shall use the following more or less standard notation. If E is an ordered vector space, $L(E)$ is the set of all linear maps $E\longrightarrow E$, and $L_+(E)$ the subset of all positivity-preserving maps. If E is a tvs as well, $\mathcal{L}(E)$ is the subset of $L(E)$ consisting of all continuous maps on E, and $\mathcal{L}_+(E)$ is the subset of all positivity-preserving continuous maps. Similarly for $L(E,F)$, $L_+(E,F)$, $\mathcal{L}(E,F)$ and $\mathcal{L}_+(E,F)$.
We shall have reasons to consider several topologies on $\mathcal{A}$. The first which we introduce now, generalizes the uniform topology for bounded operators, and was introduced by Lassner [4,47]

I.16 <u>PROPOSITION:</u> Let u be the topology of uniformly bounded convergence induced on $\mathcal{A}$ from $\mathcal{L}(W[t],W'[t'])[u]$, i.e. generated by the seminorms

$$\|b\|_{\mathfrak{M}} = \sup\{|\langle b\varphi,\psi\rangle| \ :\varphi\in\mathfrak{M}\} \qquad (38)$$

as $\mathfrak{M}$ runs through all bounded subsets of $W[t]$.

Equipped with this topology, $\mathcal{A}$ is a topological $*$-algebra, i.e. the involution is continuous and the product is separately continuous, as well as $\mathcal{A}[u]$ being a topological vector space.

As a tvs $\mathcal{A}[u]$ is incomplete. Its completion is $\mathcal{L}(W[t],W'[t'])[u] \cong W'[t'] \hat{\otimes}_\pi W'[t']$, the completed projective tensor product. Hence u is nuclear.

The topology u is determined through M by means of the equivalent family of seminorms

$$\|a\|_f = \|| f(M)\, a\, f(M) \|| \tag{39}$$

where $\||.\||$ is the operator norm on $\mathbb{B}(\mathcal{H})$, and f runs through the space

$$\{f: \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \;:\; f \text{ is bounded, continuous, and } \sup_{x\in\mathbb{R}^+} x^k f(x) < \infty \;,\; k \geq 0 \;\} \tag{40}$$

///

The statement about the completion of $\mathcal{A}[u]$ follows from the following Lemma.

I.17 <u>LEMMA:</u>

(a) $\mathcal{L}(W[t],W'[t'])[u]$ is a complete nuclear locally convex space. Moreover the following tvs isomorphism hold

$$\mathcal{L}(W[t],W'[t'])[u] \cong W'[t'] \hat{\otimes}_\pi W'[t'] \cong (W[t] \hat{\otimes}_\pi W[t])'_\beta \tag{41}$$

$$(W'[t'] \hat{\otimes}_\pi W'[t'])'_\beta \cong W[t] \hat{\otimes}_\pi W[t] \tag{42}$$

(b) $\mathcal{A}$ is dense in $\mathcal{L}(W[t],W'[t'])[u]$, hence, it is a total subset.

PROOF.

(a) is a direct consequence of I.7 and standard results on tvs theory, c.f.[37:Thm.3, p.143], [21: Thm.9, p. 496].

(b) follows because of (36), I.7 and [36:Lma. 2.6]. ∎

The following subsets of $\mathcal{A}$ are important in the sequel

I.18 DEFINITION: An element $a \in \mathcal{A}$ is said to be symmetric, or hermitian, if $a^+ = a$. The set of all hermitian elements of $\mathcal{A}$ is denoted by $\mathcal{A}_h$.

An element $a \in \mathcal{A}_h$ is said to be positive if, for all $\varphi \in W[t]$,

$$\langle a\varphi, \varphi\rangle \geqslant 0 , \tag{43}$$

and we then write $a \geqslant 0$. The set of all positive elements in $\mathcal{A}_h$ is denoted by $\mathcal{A}_+$. ///

I.19 PROPOSITION:

(a) $\mathcal{A}_h$ is a real vector subspace of $\mathcal{A}$, and $\mathcal{A} = \mathcal{A}_h + i\mathcal{A}_h$.

(b) $\mathcal{A}_+$ is a proper cone and determines a partial order in $\mathcal{A}_h$, with respect to which $\mathcal{A}_h$ is an ordered vector space: $a \geqslant b$ if and only if $a-b \in \mathcal{A}_+$, and $a = b$ if and only if $a \geqslant b$ and $b \geqslant a$.

(c) $\mathcal{A}_+$ is a normal cone in $\mathcal{A}_h$ which is generating.

(d) The order topology on $\mathcal{A}$, ρ , is given explicitly through the seminorms

$$\rho_x(a) = \sup \{|\langle a\varphi, \varphi\rangle| / \langle x\varphi, \varphi\rangle : \varphi \in W\} \quad (x \in \mathcal{A}_+), \tag{44.a}$$

where $c/0 = +\infty$ and $0/0 = 0$, $c \in \mathbb{R}^+$. Let $N_x \subset \mathcal{A}$ be the subset on which ρ_x is finite: ρ is the inductive limit topology

$$\mathcal{A}[\rho] = \lim_{x \in \mathcal{A}_+} \text{ind} \; N_x[\rho_x] \tag{44.b}$$

Then $\rho = u$ and so $\mathcal{A}[u]$ is bornological.

PROOF.

(a) It is clear that $\mathcal{A}_h$ is a real vector space. Any $a \in \mathcal{A}$ can be written as $a = a_1 + ia_2$, with $a_1 = (a+a^+)/2$ and $a_2 = (a-a^+)/2$. As a_1 and a_2 belong to $\mathcal{A}_h$, then $\mathcal{A} = \mathcal{A}_h + i\mathcal{A}_h$.

(b) Clearly $\mathcal{A}_+$ is a wedge. If $a, -a \in \mathcal{A}_+$, then $\langle a\varphi, \varphi\rangle = 0$ for all $\varphi \in W$. By the polarization identity $\langle a\varphi, \psi\rangle = 0$ for all $\varphi, \psi \in W$; hence $a=0$, and $\mathcal{A}_+$ is a proper cone [20:V.1].

(c) For all $a \in \mathcal{A}_h$, set $a = a_1 - a_2$, with $4a_1 = (I+a)^2$ and $4a_2 = (I-a)^2$, so that $\mathcal{A}_+$ is generating: $\mathcal{A}_h = \mathcal{A}_+ - \mathcal{A}_+$. For normality, see [12:4.1].

(d) The ρ-topology was introduced in [15]. Since each ρ_x is an order unit norm, ρ is the order topology [21:V.6]. That $\rho = u$ was shown by Schmüdgen [10:Cor.2 to Thm.1]. ∎

In the quantum mechanic's framework states are positive functionals (c.f. below) and the collapse of a wave packet, being a map from states to states, requires the notion of a positivity-preserving map.

I.20 PROPOSITION: A linear map $F \in L(\mathcal{A})$ is positive if $F(\mathcal{A}_+) \subset \mathcal{A}_+$; we then write $F \in L_+(\mathcal{A})$.

Every positive linear map is continuous $\mathcal{A}[u] \longrightarrow \mathcal{A}[u]$ and ρ-norm decreasing : for all $a \in \mathcal{A}$, all $x \in \mathcal{A}_+$,

$$\rho_{F(x)}(F(a)) \leqslant \rho_x(a) \qquad (45)$$

For a proof, see [15]. ///

From an abstract point of view, the set $\{\sum_{j=1}^{n} a_j^+ a_j : a_j \in \mathcal{A}, n \geqslant 1\}$ is a more natural cone for $\mathcal{A}$ than is $\mathcal{A}_+$. For applications to quantum measurement theory, however, only $\mathcal{A}_+$ is needed; hence we have an unambiguous use of the term positive. In the mathematical literature, what we call positive is rightly called strongly positive, as the above cone is always a subset of $\mathcal{A}_+$.

Let us consider next the dual of $\mathcal{A}$ and its structure. We start with some definitions.

I.21 <u>DEFINITIONS:</u>

(a) The dual of $\mathcal{A}[u]$, written $\mathcal{A}'$, is the set of all continuous linear functionals $\Phi : \mathcal{A}[u] \longrightarrow \mathbb{C}$.

A functional Φ is said to be hermitian if, for all $a \in \mathcal{A}$,

$$\overline{\Phi(a)} = \Phi(a^+) \; ; \qquad (46)$$

the set of all hermitian functionals is denoted by $\mathcal{A}_h'$.

A functional Φ is said to be positive if, for all $a \in \mathcal{A}_+$,

$$\Phi(a) \geqslant 0 \; ; \qquad (47)$$

the set of all positive fuctionals is denoted by $\mathcal{A}_+'$.

A state Φ is a positive functional which is normalized by

$$\Phi(I) = 1 \; ; \qquad (48)$$

the set of all states is denoted by $\mathbb{E}(\mathcal{A})$, or simply $\mathbb{E}$.

(b) The following subsets of $\mathbb{B}(\mathcal{H})$ are important

$$I(\mathcal{A}) = \{\rho \in \mathbb{B}(\mathcal{H}) : \text{for all } a \in \mathcal{A}, \overline{\rho a}, \overline{\rho^* a} \text{ are nuclear}\}; \qquad (49)$$

the set $I(\mathcal{A})_h$ of self-adjoint elements of $I(\mathcal{A})$, and the set $I(\mathcal{A})_+$ of positive elements of $I(\mathcal{A})$. For brevitiy we often write I, I_h and I_+. ///

The next proposition makes precise the folk-lore that all states are <density matrices>. As it is not true for states on $\mathbb{B}(\mathcal{H})$ it furnishes good evidence that the mathematical structure we have been discussing is well-adapted to quantum theory.

I.22 PROPOSITION:

(a) $I = I_h + iI_h$ and I_+ is generating for I_h: $I_+ - I_+ = I_h$. we have, further, that

$$I = \{\rho \in \mathbb{B}(\mathcal{H}) : \rho, \rho^* : \mathcal{H} \longrightarrow \mathcal{U}\}. \tag{50}$$

(b) A linear functional on $\mathcal{A}$, is a u-continuous if and only if it is of the form

$$\Phi(a) = \mathrm{tr}\,(\rho a) \qquad (\forall\, a \in \mathcal{A}) \tag{51}$$

for some $\rho \in I$. Such functionals are said to be normal.

Moreover, $\Phi \in \mathcal{A}'_+$ if and only if $\rho \in I_+$. Hence the trace determines an isomorphism between $\mathcal{A}'_+$ and I_+, $\mathcal{A}'_h$ and I_h, and $\mathcal{A}'$ and I.

(c) If $\Phi \in E$, and $\Phi(a) = \mathrm{tr}(\rho a)$, then

$$\mathrm{tr}(\rho) = 1\ , \tag{52}$$

(d) The trace is cyclic : for all $a \in \mathcal{A}$, $\rho \in I$,

$$\mathrm{tr}(\rho a) = \mathrm{tr}(a \rho) \tag{53}$$

For a proof see [9,11,16-18]. ///

In [11] Schmüdgen points out that the character $\delta_x : f \longmapsto f(x)$ is a positive functional on $\mathbb{B}(L^2(0,1))$ which is not a trace.

I.23 <u>PROPOSITION:</u> Let W be an irreducible wave function space of class $\mathcal{K}$.

The extreme states of the convex set $\mathbb{E}$ are of the form

$$\Phi(a) = tr(P_\varphi a) \quad (a \in \mathcal{A}) \tag{54.a}$$

where P_φ is the orthogonal projection onto the vector $\varphi \in W$:

$$P_\varphi(\psi) = \langle \psi, \varphi \rangle \varphi \quad (\psi \in W) \tag{54.b}$$

Hence the points of W comprise the pure states.

<u>PROOF.</u>

Let Φ be extreme. Use the GNS construction [80]. As Φ is extreme, the weak commutant of $\pi_\Phi(\mathcal{A})$ is trivial [50]. As $\pi_\Phi(\mathcal{A}), \mathcal{D}_\Phi$ satisfies the conditions of Prop.I.11, $\pi_\Phi(\mathcal{A}) \cong \mathcal{A}$ and $\mathcal{D}_\Phi \cong S^d$. ∎

Hereafter we shall identify I with $\mathcal{A}$'. for notational purposes, if $\Phi \in \mathcal{A}'$ is a linear functional, we shall write $\Phi(a) = tr(\hat{\Phi} a)$, and so $\hat{\Phi} \in I \subset \mathcal{B}(H)$ as above.

<u>THE DUAL SYSTEM ($\mathcal{A}$, $\mathcal{A}$')</u>

Our measurement theory requires an analysis of the topological structure of ($\mathcal{A}$, $\mathcal{A}$') as a dual pair. We start by proving duality, and then we introduce a number of important topologies for the pair.

I.24 <u>LEMMA:</u> The triple $[\mathcal{A}, \mathcal{A}'; \langle\ ;\ \rangle]$ is a dual pair, where

$$\langle a;\rho\rangle = tr(\rho a) \tag{55}$$

That is to say,

$$\langle a;\rho\rangle = 0 \text{ for all } \rho\in\mathcal{A}' \text{ implies } a=0\ ;$$
$$\langle a;\rho\rangle = 0 \text{ for all } a\in\mathcal{A} \text{ implies } \rho=0\ . \tag{56}$$

<u>PROOF.</u>

We only prove the first condition, since the second one is proved similarly. Let $\langle a;\rho\rangle=0$ for all ρ, in particular for $\rho_\psi(\varphi)=\langle\varphi,a\psi\rangle\psi$; all $\psi\in W$. Then $\langle a;\rho_\psi\rangle=\|a\psi\|^2$, and so $a=0$. ∎

I.25 <u>DEFINITIONS:</u> The coarsest locally convex topology on $\mathcal{A}$ compatible with the above duality is denoted $\sigma(\mathcal{A},\mathcal{A}')=\sigma$; similarly for $\sigma(\mathcal{A}',\mathcal{A})=\sigma^*$ on $\mathcal{A}'$.

The strong topologies with respect to the above duality are denoted by $\beta(\mathcal{A},\mathcal{A}')=\beta$ and $\beta(\mathcal{A}',\mathcal{A})=\beta^*$, on $\mathcal{A}$ and $\mathcal{A}'$ respectively.

The strong topologies are defined by the seminorms

$$\beta\colon a\longmapsto \sup\{|\Phi(a)|:\Phi\in\mathcal{N}\},\ \text{all weakly bounded } \mathcal{N}\subset\mathcal{A}'$$
$$\beta^*\colon a\longmapsto \sup\{|\Phi(a)|:a\in\mathcal{M}\},\ \text{all weakly bounded } \mathcal{M}\subset\mathcal{A}. \tag{57}$$

The finest locally convex topologies on $\mathcal{A}$, $\mathcal{A}'$ compatible with the duality are the Mackey topologies $\tau(\mathcal{A},\mathcal{A}')=\tau$ and $\tau(\mathcal{A}',\mathcal{A})=\tau^*$ respectively.

The structure of $\mathcal{A}'[\beta^*]$ will be useful in our theory of measurement. The following is the pertinent result.

I.26 <u>PROPOSITION:</u>

(a) For the above-defined topologies,

$$\sigma \leqslant u \leqslant \tau \leqslant \beta \qquad \text{on } \mathcal{A};$$

$$\sigma^* \leqslant \tau^* = \beta^* \qquad \text{on } \mathcal{A}' \tag{58}$$

Hence β is not generally compatible with the duality, although β^* is.

(b) The following tvs isomorphism holds:

$$\mathcal{A}'[\beta^*] \cong W[t] \hat{\otimes}_\pi W[t] \; . \tag{59}$$

The β^*-topology may be described by the norms

$$|||\Phi|||_n = |||M^n \hat{\Phi} M^n||| \qquad (n \geqslant 0) \; , \tag{60}$$

where $\Phi(a) = \mathrm{tr}(\hat{\Phi} a)$ and $|||b|||$ is the operator norm on $\mathcal{B}(\mathcal{H})$.

Hence $\mathcal{A}'[\beta^*]$ is a nuclear Frechet lmc $*$-algebra. It is barreled, bornological, complete, Mackey ($\beta^* = \tau^*$), Montel, reflexive and separable.

(c) The real subspace $\mathcal{A}_h'[\beta_h^*]$ is nuclear and Frechet, and so enjoys the above topological properties. Here $\beta_h^* = \beta^* \restriction \mathcal{A}_h'$.

(d) Let $F \in L_+(\mathcal{A})$; then F is u-continuous, and its transpose $F^t \in \mathcal{L}_+(\mathcal{A}'[\beta^*])$ exists and it is β^*-continuous.

(e) Let $G \in L_+(\mathcal{A}_h')$; then G is β_h^*-continuous.

(f) The cone $\mathcal{A}_+'$ is closed and normal in $\mathcal{A}_h'[\beta_h^*]$, and has empty interior.

PROOF.

(a) Everything is immediate from the definitions, save $\beta^* = \tau^*$ which we shall prove below.

(b) We only prove the isomorphism (59); the other assertions were proved in [18].

By I.16-17 and [20:Ex.5(c),p.116] we have

$$\mathcal{A}'[\beta^*] \cong \tilde{\mathcal{A}}'[\tilde{\beta}^*] \cong (\mathcal{L}(W[t],W'[t'])[u])'_{\tilde{\beta}^*}$$
$$\cong (W'[t']\hat{\otimes}_{\pi} W'[t'])'_{\tilde{\beta}} \cong W[t]\hat{\otimes}_{\pi} W[t] \ .$$

c.f. also [19].

(c) Nuclearity follows from the fact that $\mathcal{A}'_h$ is a subspace of $\mathcal{A}'[\beta^*]$ ([20:III.7.4]). since by (b) the involution is continuous and $\mathcal{A}'[\beta^*]$ is complete, then $\mathcal{A}'_h[\beta^*_h]$ is complete, hence closed. Therefore it is a Frechet space, because any closed subspace of a Frechet space is a Frechet space. For the reflexivity property we notice that $\mathcal{A}'_h$ is a closed subspace of the reflexive space $\mathcal{A}'[\beta^*]$ (see(b)) hence it is semireflexive. But since it is also barreled, being a Frechet space, then it is reflexive ([21:p.227]).

(d) The first part is Proposition I.20. Thus F is σ-continuous, implying that F^t exists and is σ^*-continuous [20:IV.2.1], and is obviously positive. From [20:IV.2.4] it follows that F^t is β^*-continuous.

(e) As $\mathcal{A}'_h[\beta^*_h]$ is Frechet and Mackey, and $\mathcal{A}'_+$ is generating (Proposition I.22), Then every positive linear form on $\mathcal{A}'_h[\beta^*_h]$ is continuous [20:V.5.5]; consequently, every linear map in $L_+(\mathcal{A}'_h)$ is continuous [20:V.5.6].

(f) We first show that the cone is closed. Let ρ belong to the closure of $\mathcal{A}_+'$ on the β_h^*-topology, then by (c) $\rho\in\mathcal{A}_h'$ and there exists a net ρ_α in $\mathcal{A}_+'$ converging to ρ in the β_h^*-topology. Hence it converges to ρ in the usual norm-topology of $\mathbb{B}(\mathcal{H})$ (set n=0 in I.26.b), whence ρ is positive, so $\rho\in\mathcal{A}_+'$, and so $\mathcal{A}_+'$ is closed.

We show that $\mathcal{A}_+'$ is normal by using the norms (60). For all $\Phi,\Theta\in\mathcal{A}_+'$ and all indices $n\geqslant 0$,

$$\begin{aligned}\|\!|\Phi+\Theta|\!\|_n &= \sup\{\langle(\hat{\Phi}+\hat{\Theta})M^n\varphi, M^n\varphi\rangle : \varphi\in W[t], \|\varphi\|=1\}\\ &\geqslant \sup\{\langle\hat{\Phi}M^n\varphi, M^n\varphi\rangle:\varphi\in W[t], \|\varphi\|=1\}\\ &= \|\!|\Phi|\!\|_n . \end{aligned} \tag{61}$$

thus $\mathcal{A}_+'$ is normal [20:V.3.1].

As $\mathcal{A}_h'[\beta_h^*]$ is non-normable, being an infinite dimensional nuclear space, and $\mathcal{A}_+'$ is normal, $\mathcal{A}_+'$ has no interior points [20:Ex.10(c),p.252]. ■

For later purposes we need the following technical results concerning the order properties of $\mathcal{A}'$.

I.27 <u>LEMMA:</u>

(a) For any $\Phi\in\mathcal{A}_+'$, $p\in\mathbb{N}_0$

$$\|\!|\Phi|\!\|_p = \Phi(M^{2p}) \tag{62}$$

(b) Let $\langle\Phi_n\rangle_{\mathbb{N}}$ be a monotonically increasing sequence of hermitian fuctionals $\Phi_n\in\mathcal{A}_h'$, such that for all p=0,1,2,...,

$$\lim_n \Phi_n(M^{2p}) < \infty \quad . \tag{63.a}$$

Then there exists a unique $\Phi \in \mathcal{A}_h'$ to which the sequence converges in the β^*-topology:

$$\beta^* - \lim_n \Phi_n = \Phi \qquad (63.b)$$

PROOF.

(a) $$|||\Phi|||_p = ||| M^p \hat{\Phi} M^p ||| = \| M^p \hat{\Phi} M^p \|_{tr}$$
$$= tr(M^p \hat{\Phi} M^p) = tr(\hat{\Phi} M^{2p})$$
$$= \Phi(M^{2p})$$

The first equality is just the definition of $|||\cdot|||_p$ (60), the second equality follows from the fact that $M^p \hat{\Phi} M^p \in I_+$ [3], while the latter two equalities follow from Proposition I.22.

(b) Because of (63.a), $\{\Phi_n(M^{2p}) : n \geq 1\}$ is a real Cauchy sequence for each $p \geq 0$. Introducing the signum function

$$s(n,m) = \begin{cases} +1 & \text{if } n > m \\ -1 & \text{if } n < m \end{cases} .$$

it follows that for all $n,m \geq 1$, $s(n,m)(\Phi_n - \Phi_m) \in \mathcal{A}_+'$ and

$$s(n,m)[\Phi_n - \Phi_m](a) = |[\Phi_n - \Phi_m](a)| \qquad a \in \mathcal{A}_+ .$$

we show that $(\Phi_n)_{n \geq 1}$ is a β^*-Cauchy sequence:

$$|||\Phi_n - \Phi_m|||_p = ||| s(n,m)[\Phi_n - \Phi_m] |||_p = s(n,m)[\Phi_n - \Phi_m](M^{2p})$$
$$= |[\Phi_n - \Phi_m](M^{2p})| = |\Phi_n(M^{2p}) - \Phi_m(M^{2p})| .$$

The result now follows from the completness of $\mathcal{A}_h'[\beta^*]$. ∎

I.28 COROLLARY: Let $\langle\Phi_n\rangle_{\mathbb{N}}$ be an upper bounded, monotonically increasing sequence in $\mathcal{A}_h^*$. Then there exists a $\Phi\in\mathcal{A}_h^*$ such that

$$\beta_h^* - \lim_n \Phi_n = \Phi ;$$

moreover

$$\Phi = \sup\{\Phi_n : n \geqslant 1\}. \quad ///$$

Using this result and the fact that $\mathcal{A}_h^*[\beta^*]$ is a Frechet space yields the following, c.f. also [20:V.4.3 cor(2)].

I.29 PROPOSITION: $\mathcal{A}_h^*$ is monotone complete, and β_h^* is compatible with the $\mathcal{A}_+^*$-partial order. ///

Recall that if $\Phi\in\mathbb{E}$ is a state, the spectral theorem asserts the existence of an orthonormal basis for $\mathcal{H}$, $\{e_n\in U: n \geqslant 1\}$, and sequence $\{t_n\geqslant 0: n \geqslant 1\}$ of positive reals with $\sum_n t_n=1$, such that

$$\lim_{n\to\infty} \mathrm{tr}[\hat{\Phi}-\sum_{j=1}^{n} t_j P_j] = 0. \qquad (64)$$

Here P_j is the orthogonal projection onto e_j. The relation between this expansion and the β^*-topology is this

I.30 PROPOSITION: Using the notation above,

$$\beta^* - \lim_n [\hat{\Phi}-\sum_{j=1}^{n} t_j P_j] = 0. \qquad (65)$$

PROOF.

We must show that for any index $p \in \mathbb{N}_0$

$$\lim_{n\to\infty} \left\| M^p [\hat{\Phi} - \sum_{j=1}^{n} t_j P_j] M^p \right\| = 0 \tag{66}$$

Let us abbreviate $\hat{\Phi} - \sum_{j=1}^{n} t_j P_j = \hat{\Phi}_n$. Now $\hat{\Phi}_n \in I_+$, i.e. $\hat{\Phi}_n \geq 0$ and $\hat{\Phi}_n \in I$, hence, for all $p \geq 0$ $M^p \hat{\Phi}_n M^p$ is nuclear and positive

Consequently

$$\begin{aligned} \left\| M^p \hat{\Phi}_n M^p \right\| &= \mathrm{tr}[M^p \hat{\Phi}_n M^p] \\ &= \mathrm{tr}[\hat{\Phi}_n M^{2p}] \\ &= \sum_{j \geq n+1} t_j \langle M^{2p} e_j , e_j \rangle \\ &= \sum_{j \geq n+1} t_j \left\| M^p e_j \right\|^2, \end{aligned}$$

and so (66) is true. ∎

Hereafter we shall write $\hat{\Phi} = \sum_j t_j P_j$, and the convergence is to be understood either in trace or β^*.

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SUMMARY

The pure states of a quantum mechanical system with d degrees of freedom constitute a nuclear Frechet space $W[t]$ tvs isomorphic to a Köthe sequence space of order 2. A space of class $\mathcal{K}$ constitutes an important example af a state space. Any space $W[t]$ in this class decomposes into a countable sum of irreducible wave function spaces of type S^d. Each one of this irreducible spaces is tvs isomorphic to $S(\mathbb{R}^d)$, carries a cyclic representation of the CCRs and the Fock-Cook condition is satisfied.

The algebra of observables is taken to be the complex unital $*$-algebra $\mathcal{A}=\mathcal{L}^+(W)$, equipped with one of a number of topologies: u, σ, τ, or β. It possesses a positive cone $\mathcal{A}_+$ which is normal and generating for u, and u is the order topology. With respect to u, $\mathcal{A}$ is a topological $*$-algebra.

The states are the normalized positive functionals on $\mathcal{A}$. All states are tracial (normal) and in the $\beta*$-topology, $\mathcal{A}'[\beta*]\cong W[t]\hat{\otimes}_\pi W[t]$.

Under certain conditions, positive linear maps are automatically continuous.

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CHAPTER II

OPERATIONAL APPROACH TO QUANTUM PROBABILITY

In this chapter we shall adapt to unbounded operator algebras the operational approach to quantum probability of Davies and Lewis [1].Like them we shall make an extensive use of the existing duality between the algebras of observables and states. The fact that the algebra of observables,as a topological space, is neither complete nor reflexive in the uniform topology,and the unboundedness of many of its members,confront us with several difficulties.These are more of a technical than of a conceptual nature.We overcome these difficulties by taking as a basic construction a concept introduced by Davies in [2] called an Expectation.

We now survey the chapter briefly.In sections II.1-II.5,we first,introduce the concepts of an $\mathcal{A}$ and $(\mathcal{A},W)$-measure,we then proceed to define the central notions of an expectation and instrument,and discuss their relationship. Sections II.6-II.14 are devoted to analize the topological properties of an instrument.There we show that an instrument is a bounded positive Radon Measure.Sections II.16-II.20 are concerned with the composition of instruments.There we show that in general the above operation does not yield an instrument but rather a post-instrument.Sections II.21-II.22 present the Robertson-Heisenberg uncertainty relations for observables and measurements;particular attention is payed to analyzing the statistical behaviour of measurements when certain transformations on the states are carried out.

EXPECTATIONS AND INSTRUMENTS

In the Von Neumann's scheme for quantum measurements [65],we are given a separable Hilbert space $\mathcal{H}$ and $A=\sum_{n\geq 1}\lambda_n P_n$ a bounded self-adjoint operator on $\mathcal{H}$ with eigenvalues $(\lambda_n : n\in\mathbb{N})$ and orthogonal one-dimensional projections $(P_n : n\in\mathbb{N})$ onto the corresponding eigenvectors.

A measurement of the observable A in the state Φ ,assumed normal: $\Phi(B)=tr(\hat{\Phi}B)$,will result in the ocurrence of an eigenvalue.The eigenvalues are the only allowed values that A may take.If λ_n is observed,the measurement causes the collapse of the wave packet into the pure state represented by the «density matrix» P_n.This occurs with a probability $\Phi(P_n)$.

The Davies-Lewis theory [1] adapts this scheme to symmetric operators with continuous spectra.Rather than review their work we shall point out its connection with our definitions as they occur.

In what follows we shall be using Naimark's generalization of the spectral theory.As always, $\mathcal{H}$ is a separable Hilbert space.

II.1 <u>PROPOSITION:</u> A generalized spectral family on $\mathcal{H}$ is a one-parameter family $\{B_t : t\in\mathbb{R}\}$ of operators ($0\leq B_t \leq I$) satisfying

$$\text{(i) } \mathcal{H}\text{-}\lim_{t\to-\infty} B_t = 0 \quad ; \mathcal{H}\text{-}\lim_{t\to+\infty} B_t = I \qquad (1)$$

$$\text{(ii) for all } t<s \text{ , } B_t \leq B_s \qquad (2)$$

$$\text{(iii) } \mathcal{H}\text{-}\lim_{\varepsilon\to 0^+} B_{t+\varepsilon} = B_t \qquad (3)$$

A positive operator-valued measure on $\mathcal{H}$ is a family $B : Bor(\mathbb{R}) \longrightarrow \mathbb{B}(\mathcal{H})$,where $Bor(\mathbb{R})$ is the set of all Borel subsets of $\mathbb{R}$,and satisfying

(iv) $B(\phi)=0$; $B(\mathbb{R})=I$ (4)

(v) for all $\Delta_1 \subset \Delta_2$, $B(\Delta_1) \leqslant B(\Delta_2)$ (5)

(vi) for every countable family of mutually disjoint Borel sets ,$\{\Delta_j : j \geqslant 1\}$,

$$B\Big(\bigcup_{j\geqslant 1} \Delta_j\Big)\varphi = \mathcal{H}\text{-}\lim_{n\to\infty} \sum_{1\leqslant j\leqslant n} B(\Delta_j)\varphi \qquad (6)$$

for every $\varphi \in \mathcal{H}$.

Then just as for projection-valued spectral families,every generalized spectral family determines a positive operator-valued measure and conversely.The connection is $B(t)=B((-\infty,t])$.

If (b) is a closed symmetric operator on $\mathcal{H}$,there exist at least one generalized spectral family $\{B(t):t\in\mathbb{R}\}$ such that

(vii) $D(b) \subset \{\varphi : \int_{\mathbb{R}} t^2 \langle B(dt)\varphi,\varphi\rangle < \infty\}$ (7)

(viii) for all $\varphi \in D(b)$ and all $\psi \in \mathcal{H}$,

$$\langle b\varphi,\psi\rangle = \int_{\mathbb{R}} t \,\langle B(dt)\varphi,\psi\rangle \qquad (8)$$

$$\|b\varphi\|^2 = \int_{\mathbb{R}} t^2 \langle B(dt)\varphi,\varphi\rangle \qquad (9)$$

If (b) is self-adjoint,then the family is projection-valued and unique.Moreover,equality then holds in (7).See [29,78]. ///

For a family of symmetric operators,defined on a common dense domain,we evidently require further conditions.Recall that every $b \in \mathcal{A}$ can be written as a linear combination of two symmetric operators : $\mathcal{A} = \mathcal{A}_h + i\,\mathcal{A}_h$.

II.2 <u>DEFINITIONS:</u>

(a) An $\mathcal{A}$-measure is a generalized spectral family $\{B(t): t \in R\}$ on $\mathcal{H}$ such that for all $\varphi \in W$,

$$\int_{\mathbb{R}} t\, B(dt)\varphi = b\varphi \tag{10}$$

defines a symmetric operator $b \in \mathcal{A}_h$. The integral is meant in the Riemann-Stieltjes sense, and converges in the $\mathcal{H}$-topology.

(b) An $\langle \mathcal{A}, W\rangle$-measure is an $\mathcal{A}$-measure B such that for all $\Delta \in \mathrm{Bor}(\mathbb{R})$,

$$B(\Delta)[W] \subset W \tag{11}$$

We write $\mathcal{M}_+(\mathcal{A})$,respectively $\mathcal{M}_+\langle \mathcal{A}, W\rangle$,for the set of all $\mathcal{A}$-measures, respectively $\langle \mathcal{A}, W\rangle$-measures. ///

II.3 <u>REMARK:</u>

(a) The sets $\mathcal{M}_+\langle \mathcal{A}\rangle$ and $\mathcal{M}_+\langle \mathcal{A}, W\rangle$ are convex.

(b) The inclusion $\mathcal{M}_+\langle \mathcal{A}, W\rangle \subset \mathcal{M}_+\langle \mathcal{A}\rangle$ is proper as can be seen from the coordinate multiplication operator Q on $S(\mathbb{R})$.

(c) Davies and Lewis use the term <u>observable</u> as synonymous with membership of $\mathcal{M}_+[\mathbb{B}(\mathcal{H})] = \mathcal{M}_+[\mathbb{B}(\mathcal{H}), \mathcal{H}]$. In contrast we reserve the term for membership of $\mathcal{A}$,cf. [3:3.1.1].

Aside from the continuous spectrum, two problems confront us. The first is that $\mathcal{M}_+\langle \mathcal{A}, W\rangle \neq \mathcal{M}_+\langle \mathcal{A}\rangle$,and the second is that $\mathcal{A}[u]$ is neither complete nor reflexive. This latter difficulty requires us to start our constructions by considering those linear maps on $\mathcal{A}$ which Davies calls expectations [2];by transposition we will get our notion of an instrument.

II.4 DEFINITIONS:

(a) An expectation is a map Z: $\mathrm{Bor}(\mathbb{R}) \longrightarrow L_+(\mathcal{A}_h)$ satisfying

$$\text{(i)}\quad Z(\emptyset)=0\ ,\ Z(\Delta) \geqslant 0 \quad \text{for all } \Delta\in \mathrm{Bor}(\mathbb{R}) \tag{12}$$

$$\text{(ii)}\quad \text{On } W\ ,\ Z(\mathbb{R})[I] = I \tag{13}$$

(iii) For every countable family $\{\Delta_j : j\geqslant 1\}$ of mutually disjoint Borel subsets, Z is σ-additive in the sense

$$\Phi(Z(\cup_j \Delta_j)[b])=\sum_j \Phi(Z(\Delta_j)[b])\ , \tag{14}$$

for all $\Phi\in\mathcal{A}_h'$,all $b\in\mathcal{A}$.

$$\text{(iv)}\quad \int_{\mathbb{R}} t\ Z(dt)[I] \in\mathcal{A} \tag{15}$$

where the Riemmann-Stieljes integral converges in the $\mathcal{H}$-topology.

(b) An instrument is a map $\mathcal{S}$: $\mathrm{Bor}(\mathbb{R}) \longrightarrow \mathcal{L}(\mathcal{A}_h'[\sigma^*])$ for which there is an expectation Z whose transpose satisfies

$$Z^t = \mathcal{S}\ . \qquad /// \tag{16}$$

We now consider some consequences of these definitions.The following characterization of an instrument is more or less immediate.

II.5 LEMMA: Let $\mathcal{S}$ be an instrument.Then

(i) For every $\Delta\in \mathrm{Bor}(\mathbb{R})$, $\mathcal{S}(\Delta)\in L_+(\mathcal{A}_h')$

(ii) $\mathcal{S}$: $\mathrm{Bor}(\mathbb{R}) \longrightarrow \mathcal{L}_+(\mathcal{A}_h'[\beta^*])$

(iii) For every countable family $\{\Delta_j : j \geq 1\}$ of mutually disjoint Borel sets, $\mathcal{J}$ is σ-additive in the following sense: for every $\Phi \in \mathcal{A}_h'$ and every $b \in \mathcal{A}_h$,

$$\mathcal{J}(\cup_j \Delta_j)[\Phi](b) = \sum_j \mathcal{J}(\Delta_j)[\Phi](b) \tag{17}$$

(iv) $\mathcal{J}$ preserves normalization in the sense that for every $\Phi \in \mathcal{A}_h'$,

$$\mathcal{J}(\mathbb{R})[\Phi](I) = \Phi(I) \tag{18}$$

PROOF

Properties (iii),(iv) follow from (iii),(ii) respectively of Definition II.4 .Properties (i),(ii) follow from Proposition I.26.(e). ■

TOPOLOGICAL PROPERTIES OF INSTRUMENTS

We now embark on an analysis of the topological properties of instruments. In particular, we shall show that an instrument $\mathcal{J}$ is a map $\mathrm{Bor}(\mathbb{R}) \longrightarrow \mathcal{L}_+(\mathcal{A}_h'[\beta^*])_s$ which is a bounded Radon measure. The subscript (s) indicates that $\mathcal{L}_+$ is equipped with the topology of simple convergence.

We start by showing that the σ-additive of (17) implies that $\mathcal{J}(.)[\Phi]$ is β^* σ-additive.

II.6 LEMMA: An instrument $\mathcal{J}$ is σ-additive as a map

$$\mathcal{J} : \mathrm{Bor}(\mathbb{R}) \longrightarrow \mathcal{L}_+(\mathcal{A}_h'[\beta^*])_s \tag{19}$$

PROOF.

From (i), (iii) of Lemma II.5 follows

$$0 \leqslant \mathcal{I}(\bigcup_{1\leq j\leq n} \Delta_j)[\Phi] \leqslant \mathcal{I}(\bigcup_{1\leq j\leq n+1} \Delta_j)[\Phi] \leqslant \ldots \leqslant \mathcal{I}(\bigcup_j \Delta_j)[\Phi], \quad (20.a)$$

for any positive functional $\Phi\in\mathcal{A}'_+$. By corollary I.28 it follows that

$$\beta^* - \lim_n \mathcal{I}(\bigcup_{1\leq j\leq n} \Delta_j)[\Phi] = \Psi$$

defines a functional $\Psi\in\mathcal{A}'_h$.

We know by (17) that

$$\sigma^* - \lim_n \mathcal{I}(\bigcup_{1\leq j\leq n} \Delta_j)[\Phi] = \mathcal{I}(\bigcup_j \Delta_j)[\Phi]$$

for all $\Phi\in\mathcal{A}'_h$. As $\sigma^* \leqslant \beta^*$ we see that for all $\Phi\in\mathcal{A}'_+$

$$\beta^* - \lim_n \mathcal{I}(\bigcup_{1\leq j\leq n} \Delta_j)[\Phi] = \mathcal{I}(\bigcup_j \Delta_j)[\Phi]. \quad (20.b)$$

By linearity we may extend this to all $\Phi\in\mathcal{A}'_h$, proving the assertion. ∎

The next step is to prove that $\mathcal{I}$ is inner regular.

II.7 LEMMA: An instrument $\mathcal{I}$ is inner regular: for all $\Delta\in\mathrm{Bor}(\mathbb{R})$

$$\mathcal{I}(\Delta) = s\text{-}\lim_{K\uparrow\Delta} \mathcal{I}(K) \quad (21)$$

where s is the topology of simple convergence on $\mathcal{L}(\mathcal{A}'_h[\beta^*])$ and the limit is with respect to the filtering increasing compact subsets of Δ.

PROOF.

Recall that any positive Borel measure which is finite on compact subsets of a locally compact Hausdorff space in which every open set is σ-compact is regular [25:2.18].

For any $\Phi\in\mathcal{A}_+^*$ and $p\in\mathbb{N}_0$, the set map m_p:Bor($\mathbb{R}$)$\longrightarrow\mathbb{R}$,

$$m_p(\Delta) = \mathcal{J}(\Delta)[\Phi](M^{2p})$$

is clearly a positive Borel measure on $\mathbb{R}$; and $\mathbb{R}$ is a space of the afforementioned sort. Since m_p is bounded:

$$m_p(\Delta) \leqslant m_p(\mathbb{R}) < \infty \quad ,$$

it follows that m_p is regular, hence inner regular. In virtue of the σ-compactness of $\mathbb{R}$, inner regularity can be written as [23:p.25,28]

$$m_p(\Delta) = \lim_n m_p(K_n),$$

where $(K_n:n\geqslant 1)$ is an arbitrary increasing sequence of compact subsets with $K_n\subset K_{n+1}\subset\ldots\subset\Delta$, $n\geqslant 1$, and independent of p.

Using additivity for instruments, equation (17), and the relation $B=A\cup(B\setminus A)$ for $A\subset B$, for such a sequence

$$0 \leqslant \mathcal{J}(K_n)[\Phi] \leqslant \mathcal{J}(K_{n+1})[\Phi] \leqslant \mathcal{J}(\Delta)[\Phi],$$

for any $\Phi\in\mathcal{A}_+^*$. We can now apply corollary I.28 to get

$$\mathcal{J}(\Delta)[\Phi] = \beta^* - \lim_n\mathcal{J}(K_n)[\Phi],$$

and this extends linearly to all $\Phi\in\mathcal{A}_h^*$. But this is what was to be shown, and we are done. ∎

In order that $\mathcal{J}$ be a bounded Radon measure it must certainly be bounded.

II.9 <u>LEMMA:</u> Any instrument $\mathcal{J}$ is bounded for the topology of simple convergence on $\mathcal{L}(\mathcal{A}_h^*[\beta^*])$.

PROOF.

We first notice that $\mathcal{A}$ is a total subset of $\mathcal{L}(W[t],W'[t'])[u]\cong(W[t]\hat{\otimes}_{\pi}W[t'])'_{\beta}\cong(\mathcal{A}'[\beta^*])'_{\beta}$ (Lemma I.17, Proposition I.26.b and [20:III.3.2]).

Since by Proposition I.16 the involution on $\mathcal{A}$ is continuous then $\mathcal{A}_h$ can be identified with a total subset of $(\mathcal{A}'_h[\beta^*])'_{\beta}$.

Let us show that for all $\Phi\in\mathcal{A}'_h$ the family $\{\mathcal{J}(\Delta)[\Phi]:\Delta\in Bor(\mathbb{R})\}$ is σ^*-bounded. Write $\Phi=\Phi_1-\Phi_2$ with $\Phi_1,\Phi_2\in\mathcal{A}'_+$ and let $b=b_1-b_2$ with $b_1,b_2\in\mathcal{A}_+$. Then, for all Δ,

$$\begin{aligned}|\mathcal{J}(\Delta)[\Phi](b)| &\leqslant \sum_{i,j}\mathcal{J}(\Delta)[\Phi_i](b_j)\\ &\leqslant \sum_{i,j}\mathcal{J}(\mathbb{R})[\Phi_i](b_j)\end{aligned}$$

which is finite. Hence so is $\sup\{|\mathcal{J}(\Delta)[\Phi](b)|:\Delta\in Bor(\mathbb{R})\}$, showing σ^*-boundedness.

Grothendieck has shown that if E is an F-space and $m:X\longrightarrow E$ an additive set function on a σ-algebra X, then a sufficient condition for m to be bounded is that there exist a total subset $H\subset E'$ such that $h\circ m$ is bounded for all $h\in H$ [23:Prop.0.5].But this is precisely what we have shown, with $X=Bor(\mathbb{R})$, $H=\mathcal{A}_h$, $E=\mathcal{A}'_h[\beta^*]$ and $m=\mathcal{J}(.)[\Phi]$, proving the lemma. ∎

II.9 PROPOSITION:

(a) Any instrument $\mathcal{J}$ is a continuous mapping

$$\mathcal{J}:\mathbb{B}(\mathbb{R})\longrightarrow\mathcal{L}(\mathcal{A}'_h[\beta^*])_s$$

where $\mathbb{B}(\mathbb{R})$ is the normed space of real valued bounded Borel functions equipped with the supremum norm. Thus $\mathcal{J}$ is a bounded Radon mapping in the sense of Thomas [23].

(b) For every $\Phi\in\mathcal{A}_h^*$, $\mathcal{J}(.)[\Phi]$ is a bounded Radon measure.

PROOF.

The first part is a consequence of Lemmas (II.6-II.8) and [43].

For the second part we note that $\mathcal{A}_h^*[\beta*]$ is reflexive, and then apply Theorem 5.3 (p.136) and Remark (5.8)(p.139) of [43]. ∎

We sharpen this result, obtaining the desired property of instruments.

II.10 THEOREM: Any instrument $\mathcal{J}$ is a bounded Radon measure

$$\mathcal{J} : \mathrm{Bor}(\mathbb{R}) \longrightarrow \mathcal{L}_+(\mathcal{A}_h^*[\beta*])_s \tag{22}$$

PROOF.

As $\mathcal{A}_h^*[\beta*]$ is complete, the completion of $\mathcal{L}(\mathcal{A}_h^*[\beta*])_s$ is $L(\mathcal{A}_h^*[\beta*])_s$ ([37:p.144]).

Now $W[t]$ has a countable basis, say $\{e_n;n\in\mathbb{N}\}$ (I.7.c). By (I.26.b) $\mathcal{A}_h^*[\beta*]\cong W[t]\hat{\otimes}_\pi W[t]$, hence it has a countable linear base; namely one isomorphic to $\{e_n\otimes e_m:n,m\in\mathbb{N}\}$, c.f.[21:p.23].

It follows that

$$L(\mathcal{A}_h^*[\beta*])_s = \prod_{\mathbb{N}} \mathcal{A}_h^*[\beta*] \tag{23}$$

with the product topology ([37:p.134]). Thus $L(\phi_h^{\prime}[\beta^*])_s$ is reflexive and Frechet [20:II.4. p.49,IV.5.8].

By [20:V.5.1] $\mathcal{L}_+(\phi_h^{\prime}[\beta^*])$ is a closed subspace of $\mathcal{L}(\phi_h^{\prime}[\beta^*])_s$ and thus of $L(\phi_h^{\prime}[\beta^*])$. Consequently $\mathcal{L}_+(\phi_h^{\prime}[\beta^*])$ is reflexive and Frechet because any closed subspace of a Frechet-reflexive space is itself a Frechet-reflexive space [20:II.4 p.49,IV.5.5] (see also I.26.c). We then proceed exactly as on the proof of II.9. ∎

II.11 <u>COROLLARY:</u>

(a) $L_+(\phi_h^{\prime}[\beta^*]) = \mathcal{L}_+(\phi_h^{\prime}[\beta^*])$

(b) $\mathcal{L}_+(\phi_h^{\prime}[\beta^*])$ is an proper cone in $\mathcal{L}(\phi_h^{\prime}[\beta^*])_s$

(c) $\mathcal{L}(\phi_h^{\prime}[\beta^*])$ and $\mathcal{L}_+(\phi_h^{\prime}[\beta^*])_s$ are nuclear.

<u>PROOF.</u>

(a) now follows from Proposition I.26.

(b) is true because $\phi_+^{\prime}$ is total in $\phi_h^{*}[\beta^*]$, c.f.[20:Prop.5.1 p.226].

(c) follows from the product representation eqn (23) above for $L(\phi_h^{\prime}[\beta^*])_s$ and the fact that $\phi_h^{\prime}[\beta^*]$ is nuclear [21:Cor(3) p.483]. ∎

Our next proposition brings a degree of physical interpretation by proving that an instrument uniquely determines an (ϕ,W)-measure. The reverse implication is one of non-uniqueness: a given (ϕ,W)-measure is determined by many instruments. This is consonant with experience: there are many ways to measure the position of a particle.

II.12 <u>PROPOSITION:</u> Given an instrument $\mathcal{S}$ there is a unique $(\mathcal{A},\mathcal{W})$-measure $\Delta \longmapsto m(\mathcal{S};\Delta)$ determined by it. $m(\mathcal{S})$ is given by extension from $\mathcal{W}$ to $\mathcal{H}$ of

$$m(\mathcal{S};\Delta)\varphi = Z(\Delta)[I]\varphi \quad ,(\varphi\in\mathcal{W}) \qquad (24)$$

where Z is the expectation, unique , for which $Z^t=\mathcal{S}$. Hence, for any state Φ

$$\Phi[\,m(\mathcal{S};\Delta)] = (\mathcal{S}(\Delta)\Phi)[I]$$
$$= \Phi[Z(\Delta)[I]] \,. \qquad (25)$$

Given any $(\mathcal{A},\mathcal{W})$-measure B, there exists many instruments $\mathcal{S}$ such that $m(\mathcal{S};\Delta)=B(\Delta)$. For example, for each state Φ, the instrument

$$\mathcal{S}(\Phi;\Delta) = Z(\Phi;\Delta)^t \;;$$
$$(b\in\mathcal{A},\ \Delta\in Bor(\mathbb{R})) \qquad (26)$$
$$Z(\Phi;\Delta)[b] = \Phi(b)B(\Delta)$$

is of this sort.

<u>PROOF.</u>

From the definition of an expectation it follows that for any $\Delta\in Bor(\mathbb{R})$, $m(\mathcal{S};\Delta)[\mathcal{W}]\subset\mathcal{W}$ and $0 \leqslant m(\mathcal{S};\Delta) \leqslant I$ and so an extension of domain to $\mathcal{H}$ is posible. For brevity we drop the $\mathcal{S}$ temporarily.

In the first place m must be shown to be a generalized spectral family. Let $\{\Delta_i\in Bor(\mathbb{R}): i\in\mathbb{N}\}$ be a sequence of mutually disjoint Borel sets and let P be the bounded $\mathcal{H}$-operator $P(h)=\langle h,\varphi\rangle\psi$, $\varphi,\psi\in\mathcal{W}[t]$ arbitrary. Taking Φ to be the functional determined by P and with $b=I$, the additivity (14) of Z yields

$$\langle m(\bigcup_j \Delta_j)\varphi,\psi\rangle = \sum_j \langle m(\Delta_j)\varphi,\psi\rangle \qquad (27)$$

We now apply the following theorem of Thomas [23]: if $M:Bor(\mathbb{R})\longrightarrow\mathcal{H}$ is a set function such that $\psi\circ M$ is σ-additive for all $\psi\in W$, then M is σ-additive. Here we view W as a subset of the dual of $\mathcal{H}$. With $M(\Delta)=m(\Delta)\varphi$, eqn (27) implies that $\overset{m(.)\varphi}{\vee}$ is σ-additive.

As $m(\Delta)$ is bounded, we can extend (27) to $\psi\in\mathcal{H}$ and transpose $m(\Delta)$ to act on ψ. Repeating the above argument then implies that $m(\Delta)\varphi$ if σ-additve for all $\varphi\in\mathcal{H}$. Hence m is a generalized spectral measure.

Equation (15) now implies that m is an $(\mathcal{A},W)$-measure.

Equation (25) follows immediately from (24) and the spectral decomposition I.64 .

Finally, the assertion that $\mathcal{J}(\Phi;.)$ is an instrument which determines $B(.)$ is obvious. ∎

Following Davies [2] we introduce the class of observables which can be measured. By this we mean that there exists at least one instrument which will give some information about the observable in question.

II.13 <u>DEFINITIONS:</u>

(a) Given an $\mathcal{A}$-measure B, let $\mathcal{F}(B)$ be the following set of functions of B:

$$\mathcal{F}(B) = \sigma[\mathbb{B}(\mathcal{H})_h, \mathcal{A}_h']\text{-}\overline{\vee\{B(\Delta): \Delta\in Bor(\mathbb{R})\}}, \qquad (28.a)$$

where $\vee$ indicates linear span, and the closure of the span is in the indicated topology. Note that $\mathbb{B}(\mathcal{H})_h$, and $\mathcal{A}_h'$ constitute a dual pair.

We introduce a partial order on $\mathcal{M}_+(\mathcal{A})$ by setting

$$B \prec C \quad \text{if } \mathcal{F}(B) \subset \mathcal{F}(C) \tag{28.b}$$

Following [2], we say that B give less information than C.

(b) An $\mathcal{A}$-measure $B \in \mathcal{M}_+(\mathcal{A})$ is said to be physical if there exists an $(\mathcal{A},W)$-measure $\mathcal{A} \in \mathcal{M}_+(\mathcal{A},W)$ giving less information than B, i.e. $A \prec B$.

(c) An observable $b \in \mathcal{A}$ is said to be physical if and only if it has a spectral representation by a physical $\mathcal{A}$-measure B. If $A \prec B$ and A is the spectral representation of the observable $a \in \mathcal{A}$, we say that a is a regularization of b.

(d) If $b \in \mathcal{A}$ is physical, and $B \in \mathcal{M}_+(\mathcal{A})$ is any physical $\mathcal{A}$-measure spectrally representing b, then any instrument $\mathcal{S}$ such that $m(\mathcal{S};.) \prec B$ is said to be an instrument for measuring b. ///

II.14 <u>INTERPRETATION:</u> Let $b \in \mathcal{A}$ be a physical observable, and $\mathcal{S}$ an instrument for measuring it. For any state Φ, the probability of obtaining an observation in $\Delta \in Bor(\mathbb{R})$ on $\mathcal{S}$ is

$$\Phi[M(\mathcal{S};\Delta)] = pr(\mathcal{S};\Phi;\Delta) \tag{29}$$

If such an observation occurs, the state Φ collapses to the state

$$\Phi \longmapsto \mathcal{S}(\Delta)[\Phi]/\mathcal{S}(\Delta)[\Phi](I) \tag{30}$$

II.15 <u>REMARKS:</u>

(a) An open question which would be of some interest to answer is the relation between physicality and the structure of $\mathcal{M}_+(\mathcal{A})$. In particular do non-physical observables exists?; when, if ever do, a regularization of an observable exist which is maximal with respect to the information-partial order?

(b) Let $b=\sum_n \lambda_n P_n \in \mathcal{A}$ be self-adjoint such that $P_n[W]\subset W$ for all $n \geqslant 1$. The associated spectral measure P is an $(\mathcal{A},W)$-measure such that $P(\Delta)=\sum P_n (\lambda_n \in \Delta)$. A (best) instrument for measuring b is given by

$$\mathcal{S}(\Delta)[\Phi] = \sum \Phi(P_n \cdot P_n) \quad (\lambda_n \in \Delta). \tag{31}$$

This is the familiar collapse formula for the discret case.

Of course the instrument $\mathcal{S}$ is also an instrument for measuring other observables, e.g., $a=b+c$ for $c\in\mathcal{A}$, c possesses a spectral $\mathcal{A}$-measure C with support disjoint from $\{\lambda_n : n \geqslant 1\}$. In this case our experiment obtains no information about the part of a determined by c.

(c) If $B\in\mathcal{M}_+(\mathcal{A})$ is physical and projection valued, then any regularization of B is an abelian family.

COMPOSITION AND CONDITIONING

Let $\mathcal{S}_1, \mathcal{S}_2$ be instruments. Suppose that we measure with $\mathcal{S}_1$ on a state Φ and obtain a positive result in the Borel region Δ_1; then we immediately measure with $\mathcal{S}_2$ in the new state. If we get a positive result in the region Δ_2 the final outcome will be the normalized form of

$$\mathcal{S}_{12}(\Delta_2\times\Delta_1)[\Phi] = \mathcal{S}_2(\Delta_2)(\mathcal{S}_1(\Delta_1)[\Phi]) \tag{32}$$

By virtue of our constructions $\mathcal{S}_{12}(\Delta_2\times\Delta_1)[\Phi]$ is a positive functional for all Borel rectangles $\Delta_2\times\Delta_1$ and all positive functionals Φ. Moreover it is the transpose of an $\mathcal{A}$-stable map. However, we shall now show that whilst there is a unique extension to all Borel sets in $Bor(\mathbb{R}^2)$, the extension is not in general an instrument. Rather it is the transpose of a map with range in $\tilde{\mathcal{A}} \cong W'[t']\hat{\otimes}_\pi W'[t']$, the completion of $\mathcal{A}$ in the u topology (c.f. Prop. I.16).

II.16 <u>PROPOSITION:</u> Let $\mathcal{S}_{12}$ be defined as in eqn (32) above. There exists a unique inner regular Radon measure

$$\mathcal{S}_{12} : Bor(\mathbb{R}^2) \longrightarrow \mathcal{L}_+(\mathcal{A}'[\beta^*]_h)_s$$

such that for all Borel rectangles expresion (32) results.

In general $\mathcal{S}_{12}$ is the transpose of a map

$$Z_{12} : Bor(\mathbb{R}^2) \longrightarrow \mathcal{L}_+(\tilde{\mathcal{A}}_h[u])$$

and is not, therefore, an instrument. ///

Before proving this proposition we present two preliminary lemmas. Note firt that we have taken the liberty of implicity extending all previous definitions and constructions of this chapter, from $Bor(\mathbb{R})$ to $Bor(\mathbb{R}^2)$. Obviously all the results remain true.

The measure $\mathcal{S}_{12}$ will be referred to as the compose of $\mathcal{S}_2$ and $\mathcal{S}_1$.

II.17 <u>LEMMA:</u> Let $\{T_\alpha : \alpha \in J\}$ be an upper bounded upward directed subset of $\mathcal{L}_+(\mathcal{A}_h^*[\beta_h^*])_s$. then there exists $T \in \mathcal{L}_+(\mathcal{A}_h^*[\beta_h^*])_s$ so that

$$\begin{aligned} T &= \sup\{T_\alpha : \alpha \in J\} \\ &= s - \lim_J T_\alpha \qquad (33) \end{aligned}$$

PROOF.

By proposition I.29, $\{T_\alpha\Phi : \alpha\in J\}$ converges to its supremum for each $\Phi\in\mathcal{A}_+'$. As $\mathcal{L}_+(\mathcal{A}_h'[\beta_h^*])_s$ is complete the result follows. ∎

II.18 LEMMA: For any $S,T\in\mathcal{L}_+(\mathcal{A}_h'[\beta_h^*])$ let us define

$$\langle S,T\rangle = ST \quad . \tag{34}$$

Then $\langle\ ,\ \rangle$ is a bipositive bilinear map such that whenever $\{T_\alpha : \alpha\in J\}$ is a net as in Lemma II.17

$$\langle S,T\rangle = \text{s-}\lim_J \langle S,T_\alpha\rangle \tag{35}$$

$$\langle T,S\rangle = \text{s-}\lim_J \langle T_\alpha ,S\rangle \tag{36}$$

for all S.

Hence $\langle\ ,\ \rangle$ is a complete proper bipositive bilinear map in the sense of [44].

PROOF.

It is clear that $\langle\ ,\ \rangle$ is a bipositive bilinear map. Since $T=\text{s-}\lim T_\alpha$ and S belong to $\mathcal{L}_+(\mathcal{A}_h'[\beta_h^*])_s$ then

$$(TS)[\Phi]=T(S[\Phi])= \beta^*\text{-}\lim T_\alpha(S[\Phi])= \beta^*\text{-}\lim_J (T_\alpha S)[\Phi] \tag{37.a}$$

and

$$(ST)[\Phi]=S(T[\Phi])= \beta^*\text{-}\lim_J S(T_\alpha[\Phi])= \beta^*\text{-}\lim_J (ST_\alpha)[\Phi] \tag{37.b}$$

for any $\Phi\in \mathcal{A}_h'[\beta_h^*]$.

Equations (37.a) and (37.b) now imply

$TS=\text{s-}\lim T_\alpha S$ and $ST=\text{s-}\lim ST_\alpha$ respectively,

from which the assertion of the Lemma follows. ∎

PROOF of PROPOSITION II.16

The existence of a unique extension of δ_{12} to $\mathrm{Bor}(\mathbb{R}^2)$, satisfying eqn (32) follows from applying Lemma II.18 to Theorem (1.7) of [44]. Hence, for any $\Delta \in Bor(\mathbb{R}^2)$, $\mathcal{S}_{12}(\Delta) \in \mathcal{L}_+(\mathcal{A}_h^*[\beta_h^*])$; but this implies $\mathcal{S}_{12}(\Delta) \in \mathcal{L}_+(\mathcal{A}_h^*[\sigma_h^*])$. thus there exists the adjoint $[\mathcal{S}_{12}(\Delta)]^t = Z_{12}(\Delta)$ belonging to $\mathcal{L}_+(\tilde{\mathcal{A}}_h[\sigma])$, so that, for any $\Delta_1, \Delta_2 \in Bor(\mathbb{R})$ $Z_{12}(\Delta_1 \times \Delta_2) \in \mathcal{L}_+(\mathcal{A}_h[u])$; the latter assertion follows from (32), the fact that $\mathcal{S}_1(\Delta_1), \mathcal{S}_2(\Delta_2) \in \mathcal{L}_+(\mathcal{A}_h^*[\sigma^*])$ and Proposition I.20, while the first follows because $(\mathcal{A}_h^*[\beta^*])^*$ is isomorphic to $\tilde{\mathcal{A}}_h$. [20:IV.7.4] now implies that $Z_{12}(\Delta) \in \mathcal{L}_+(\tilde{\mathcal{A}}_h[u])$, as was to be shown. ∎

The fact that $\mathcal{S}_{12}$ is not an instrument is mildly disturbing. However, a strong case can be made, on operational grounds, for defining the expectations and instruments not on all Borel sets, but only on intervals, perhaps only on finite intervals. It is difficult to imagine, e.g. how one would measure the position of a particle within an extremely wild Borel set, nor even why one would wish to. Be that as it may, it seems useful to introduce the notions of pre and post instruments and epectations.

II.19 <u>DEFINITIONS:</u>

(a) A pre-expectation is a map $Z: \mathbb{P}^d \longrightarrow L_+(\mathcal{A}_h)$ satisfying the conditions i-iv of an expectation (Definition II.4) save that the $\mathbb{P}^d$ is the ring generated by all polyintervals in $\mathbb{R}^d$, so the families of disjoint sets must satisfy $\Delta_j \in \mathbb{P}^d$ and $\cup \Delta_j \in \mathbb{P}^d$. As well, the integrals eqn (15) is over $\mathbb{P}^d$.

A post-expectation is a map $Z:\mathrm{Bor}(\mathbb{R}^2)\longrightarrow L_+(\tilde{A}_h)$ satisfying conditions i-iii, with iv replaced by

$$\int_{\mathbb{R}^d} t_1 t_2 \ldots t_d Z(dt_1, dt_2, \ldots, dt_d)[I] \in \tilde{A}_h \qquad (16.b)$$

(b) A pre-instrument is a map $\mathcal{S}: \mathbb{P}^d \longrightarrow \mathcal{L}(A_h^*[\sigma*])$ which is the transpose of a pre-expectation.

A post-instrument is a map $\mathcal{S}:\mathrm{Bor}(\mathbb{R}^d)\longrightarrow \mathcal{L}(A_h^*[\sigma*])$ which is the transpose of a post-expectation. ///

We shall not examine the consequences of this definitions here save for three remarks. The first is that we could have demanded only finite additivity for pre-expectations and pre-instruments. However one could always extend uniquely to σ-additivity, c.f.[79] and references therein.

The second remark is that every pre-instrument has a unique continuous extension to a post-instrument. The quality of being an instrument seems delicate.

Thirdly every instrument appearing in this paper composes with every other such instrument to give an instrument. We take this to be result of special circumstances. Nonetheless, we do not have a counter-example.

Davies and Lewis also defined joint distributions and conditioned observables.

II.20 <u>PROPOSITION:</u> Let Z_1, Z_2 be expectations, B_1, B_2 their respective (A,W)-measures, and $\ ,\ \mathcal{S}_2$ the respective instruments.

The joint distribution of $\mathcal{J}_2$ following $\mathcal{J}_1$ is defined to be the map

$$Z_{12} : \mathrm{Bor}(\mathbb{R}^2) \longrightarrow \mathcal{L}_+(\tilde{\mathcal{A}}_h[u]); \quad Z_{12}(\omega) = \mathcal{J}_{12}^{t}(\omega).$$

Then Z_{12} is a post-expectation whose marginal distributions satisfy

$$Z_{12}(\mathbb{R}\times\Delta) = Z_1(\mathbb{R})Z_2(\Delta); \; Z_{12}(\Delta\times\mathbb{R}) = Z_1(\Delta)Z_2(\mathbb{R}),$$

Hence

$$Z_{12}(\mathbb{R}\times\Delta)[I] = Z_1(\mathbb{R})B_2(\Delta); \; Z_{12}(\Delta\times\mathbb{R})[I] = B_1(\Delta),$$

are $(\mathcal{A},W)$-measures.

The map $\Delta \longmapsto Z_1(\mathbb{R})B_2(\Delta)$ is the $(\mathcal{A},W)$-measure B_2 conditioned by the measurement of B_1 with the instrument $\mathcal{J}_1$. c.f.[1:Theorem 3]. ///

We note here that instruments generally have no repeatibility properties. We have not examined the ε-repeatibility properties of our instruments, c.f.[1].

UNCERTAINTY RELATIONS

Let B(.) be an $\mathcal{A}$-measure, for any state $\Phi \in \mathbb{E}$

$$\mu_\Phi(dt) = \Phi(B(dt)) \tag{38.a}$$

defines a probability measure on the real line. By means of this distribution we now define two important characteristics of B(.)

II.21 <u>DEFINITION:</u> The mean value $E_\Phi(B)$ and the variance $D_\Phi(B)$ of the $\mathcal{A}$-measure B(.) in the state Φ are defined as the mean value and the variance of the distribution $\mu_{\Phi}(dt)$ respectively, i.e.

$$E_{\Phi}(B) = \int_{\mathbb{R}} t\, \mu_{\Phi}(dt),$$
$$D_{\Phi}(B) = \int_{\mathbb{R}} (t-E_{\Phi}(B))^2 \mu_{\Phi}(dt) \,. \qquad (38.b)$$

These quantities are well defined if $\mu_{\Phi}(dt)$ has finite second moment. In that case we shall say that the $\mathcal{A}$-measure $B(.)$ has finite second moment.

II.22 <u>PROPOSITION:</u> Any $\mathcal{A}$-measure $B(.)$ has finite first and second moment with respect to any state $\Phi \in \mathbb{E}$.

<u>PROOF.</u>

By the spectral decomposition of $\hat{\Phi}$ (Prop.I.30), $\hat{\Phi} = \sum_{j\geq 1} s_j P_j$ with $s_j \geq 0$, $\sum_{j\geq 1} s_j = 1$ and $P_j = \langle\ ,e_j\rangle e_j$, $\mathcal{H} \supset \{e_j : j \geq 1\}$ an orthonormal basis. We then have

$$\begin{aligned}\int_{\mathbb{R}} t^2\, \Phi(B(dt)) &= \int_{\mathbb{R}} t^2\ \mathrm{tr}(\hat{\Phi}\, B(dt)) = \int_{\mathbb{R}} t^2 \left(\sum_{j\geq 1} s_j \langle B(dt)e_j, e_j\rangle\right) \\ &= \sum_{j\geq 1} s_j \int_{\mathbb{R}} t^2 \langle B(dt)e_j, e_j\rangle = \sum_{j\geq 1} s_j \|be_j\|^2 \\ &= \mathrm{tr}(\hat{\Phi}\, b\ ) \\ &= \Phi(b^2) < \infty \end{aligned} \qquad (39)$$

Here $b = \int_{\mathbb{R}} tB(dt) \in \mathcal{A}_h$ is the observable determined by $B(.)$. The first equality is clear, the second one follows from the boundedness of $B(dt)$ and the above spectral decomposition of $\hat{\Phi}$. We have used positiveness in the third equality, Eqn (9) in the fourth and the fact that $b \in \mathcal{A}_h$ in the latter two equalities.

To show that $B(.)$ has finite first moment we notice that

$$\left|\int_{\mathbb{R}} t\,\mu_{\Phi}(dt)\right|^2 \leqslant \int_{\mathbb{R}} t^2\mu_{\Phi}(dt), \tag{40}$$

from which the desired result follows. ∎

Now by Proposition II.1 to any $b\in\mathcal{A}_h$ there corresponds a (generally, non-unique) $\mathcal{A}$-measure $B(.)$ such that for any $\varphi\in W$

$$b\varphi = \int_{\mathbb{R}} tB(dt)\varphi \quad .$$

The quantities (38.b) are the same for all $\mathcal{A}$-measures $B(.)$ of b: for a pure state, this follows directly form Eqns (8),(9); while for a general state $\Phi\in\mathbb{E}$ the result follows from the spectral decomposition I.30 and the above case.

Therefore, we can call Eqns (38.b) correspondingly, the mean $E_{\Phi}(b)$ and the variance $D_{\Phi}(b)$ of the observable b. In virtue of Eqns (8) and (9), for a pure state $\Phi \longleftrightarrow \hat{\Phi}=P_{\varphi}$ $(\varphi\in W)$, we have

$$E_{\Phi}(b) = \langle b\,\varphi,\varphi\rangle$$

$$D_{\Phi}(b) = \|(b-E_{\Phi}(b)I)\varphi\|^2 = \|b\varphi\|^2 - E_{\Phi}(b)^2 \ . \tag{41}$$

If $c\in\mathcal{A}_h$ is another observable, then for any real t

$$0 \leqslant \|(b-E_{\Phi}(b)I)\varphi - it(c-E_{\Phi}(c)I)\varphi\|^2$$

$$= D_{\Phi}(b)+tE_{\Phi}(i[b,c])+t^2D_{\Phi}(c), \tag{42}$$

hence

$$D_{\Phi}(b)D_{\Phi}(c) \geqslant (\tfrac{1}{4})|E_{\Phi}(i[b,c])|^2 \quad , \tag{43}$$

where

$$[b,c] = bc - ca \tag{44}$$

is the commutator of the operators b and c. Notice that as $b,c\in\mathcal{A}_h$ then $i[b,c]\in\mathcal{A}_h$.

The inequality (44) is called the UNCERTAINTY RELATION. We shall now generalize this inequality to any state $\Phi\in\mathbb{E}$ and $\mathcal{A}$-measure $B(.)$.

We first notice that from the las three equalities of (39), for any $b \in \mathcal{A}_h$, $\Phi \in \mathbb{E}$

$$\Phi(b^2) = \sum_{j \geq 1} s_j \, \| b e_j \|^2 < \infty \quad , \tag{45}$$

where Φ is decomposed as in the proof of proposition II.21. The operators belonging to $\mathcal{A}_h$ are not the only ones satisfying (45), for instance any $h \in \mathbb{B}(\mathcal{H})$ also satisfies this condition. They belong to a special class of operators; the so called square summable operators, which were introduced by Holevo, c.f. [66,81,82]

II.23 <u>DEFINITION:</u> Using the notation above, any symmetric operator b on $\mathcal{H}$ satisfying equation (45) is called a square-summable operator with respect to Φ. Two such operators b_1, b_2 are equivalent if

$$b_1 e_j = b_2 e_j \qquad \text{for } s_j = 0 \tag{46}$$

By $L^2(\Phi)$ we shall denote the set of equivalence classes of square-summable operators with respect to Φ endowed with the inner product.

$$\langle a, b \rangle_\Phi = \mathrm{Re}\,\Phi(ab). \tag{47}$$

///

Some important consequences of these definitions are collected in the following proposition.

II.24 <u>PROPOSITION:</u> Let $\Phi \in \mathbb{E}$.

(a) $L^2(\Phi)$ is a Hilbert space.

(b) For any $\mathcal{A}$-measure $B(.)$ and $f \in L^2(\mathbb{R}, \Phi(B(dt)))$

$$\int_{\mathbb{R}} f(t) B(dt) \in L^2(\Phi) \tag{48}$$

and

$$\Phi([\int_{\mathbb{R}} f(t)B(dt)]^2) \leqslant \int_{\mathbb{R}} |f(t)|^2 \Phi(B(dt)) \qquad (49)$$

(c) $\mathcal{A}_h \subset \bigcap_{\mathbb{E}} L^2(\Phi)$ (50)

PROOF.

(a) and (b) were proven by Holevo: c.f.[66,67,81,82] and references therein.

(c) follows from equation (45). ∎

We are now ready to establish a general version of inequality (44)

II.25 PROPOSITION: For any state Φ and $a,b \in \mathcal{A}_h$

(i) $E_\Phi(b) = \Phi(b)$ (51)

(ii) $D_\Phi(b) = \Phi([b-E_\Phi(b)I]^2)$ (52)

(iii) $\Phi(a^2) + \Phi(b^2) \geqslant \Phi(i[a,b])$ (53)

(iv) $\Phi(a^2)\Phi(b^2) \geqslant (\frac{1}{4})|\Phi(i[a,b])|^2$ (54)

For any $\mathcal{A}$-measure $B(.)$ with corresponding observable b,

(v) $E_\Phi(B) = \Phi(b)$ (55)

(vi) $D_\Phi(B) \geqslant \Phi([b-E_\Phi(B)]^2)$ (56)

Moreover, if $C(.)$ is another $\mathcal{A}$-measure with corresponding observable c, then

(vii) $D_\Phi(B)D_\Phi(C) \geqslant (\frac{1}{4})|\Phi(i[b,c])|^2$ (57)

PROOF.

(i) and (ii) follows from (41), the spectral decomposition of Φ (Proposition I.30) and Proposition II.22.

To show (iii) we notice that

$$(a-ib)^*(a-ib) = a^2-iab+iba+b^2 \geqslant 0 \qquad (58)$$

hence

$$a^2+b^2 \geqslant i(ab-ba) = i[a,b] \qquad (59)$$

thus, by the positivity of Φ and the fact that $i[a,b]\in\mathcal{A}_h$

$$\Phi(a^2+b^2) \geqslant \Phi(i[a,b]) \qquad (60)$$

which establishes (iii).

Putting ta ($t\in\mathbb{R}$) in place of a in (53) we obtain

$$t^2\,\Phi(a^2) - t\,\Phi(i[a,b]) + \Phi(b^2) \geqslant 0 \qquad (61)$$

from which inequality (54) follows.

(v) follows immediately from the expansion

$$E_\Phi(B)=\int_{\mathbb{R}} t\,\mu_\Phi(dt)=\sum_{j\geqslant 1} s_j \int_{\mathbb{R}} t\langle B(dt)e_j,e_j\rangle=\sum_{j\geqslant 1} s_j\langle be_j,e_j\rangle$$
$$=\Phi(b) \qquad (62)$$

(vi) is obtained by putting $f(t)=t-E_\Phi(B)$ in Proposition II.24.

Finally, to show (vii) we use inequalities (56) and (54) to obtain

$$D_\Phi(B)D_\Phi(C) \geqslant \Phi([b-E_\Phi(B)]^2)\,\Phi([c-E_\Phi(C)]^2)$$
$$\geqslant (\tfrac{1}{4})|\Phi(i[b-E_\Phi(B),c-E_\Phi(C)])|^2$$
$$= (\tfrac{1}{4})|\Phi(i[b,c])|^2 . \qquad \blacksquare$$

We remark here that inequality (57) in contrast to (54), applies also to joint measurements when $B(dt)$ and $C(dt)$ are marginal measurements with respect to some $M(dt\ ds)$.

Let us now analize the form of the uncertainty inequality (57) for a special class of $\mathcal{A}$-measurements.

First, we introduce the following definition

II.26 DEFINITION: A one-parameter $\mathcal{A}$-unitary group U_t $(t\in\mathbb{R})$ is a one parameter unitary group in $\mathcal{H}$ so that

(i) its self-adjoint infinitesimal generator $\langle a\rangle$ belongs to $\mathcal{A}_h$,

(ii) $U_t \in \mathcal{A}$, $t \in \mathbb{R}$. ///

Now, assume that a state Φ_0 of a quantum mechanical system is prepared by a device to which a frame of reference is related. Furthermore, suppose that a transformation U_t of the frame of reference, which is represented in $\mathcal{H}$ by an one-parameter $\mathcal{A}$-unitary group $\hat{U}_t=e^{ita}$ $(t\in\mathbb{R})$, is carried out. Then the change of state of the system under this transformation is given by

$$\Phi_t = U_t \Phi_0 U_{-t} \tag{63.a}$$

or equivalently

$$\begin{aligned}\hat{\Phi}_t &= \hat{U}_t \hat{\Phi}_0 \hat{U}_{-t} \\ &= e^{ita} \hat{\Phi}_0 e^{-ita}\end{aligned} \tag{63.b}$$

II.27 EXAMPLES: Let us take $\mathcal{A} = \mathcal{L}^+(S(\mathbb{R}^d))$.

(a) If the frame of reference is shifted along some axis a distance t, then a=P the momentum operator along this axis.

(b) If the frame of reference is rotated around an axis by the angle θ=t, then a=J, the angular momentum for the axis.

(c) If the origin of the time reference is shifted by the quantity t, then a=-H, the Hamiltonian of the particle, provided the latter is an S-Hamiltonian (see next chapter def. III.5). ///

On the states given by Eqn(63.a), the inequality (57) takes the following form

II.28 PROPOSITION: Using the notation above, if $B(.)$ is an $\mathcal{A}$-measure then

$$D_{\Phi_t}(B)D_{\Phi_t}(a) \geq (\tfrac{1}{4})[(d/dt)E_{\Phi}(B)]^2 \tag{64}$$

PROOF.

From (56, (62) and (54) we get

$$D_{\Phi_t}(B)D_{\Phi_t}(a) \geq (\tfrac{1}{4})|\Phi_t(i[b,a])|^2 \tag{65}$$

where b is the observable corresponding to $B(.)$

Now

$$E_{\Phi_t}(B) = \int_{\mathbb{R}} s\,\Phi_t(B(ds)) = \int_{\mathbb{R}} s\ \mathrm{tr}(\hat{\Phi}_t(B(ds))) \tag{66}$$

But

$$\mathrm{tr}(\hat{\Phi}_t(B(ds)) = \mathrm{tr}(e^{ita}\hat{\Phi}_o e^{-ita}B(ds))$$

$$= \mathrm{tr}(\hat{\Phi}_o e^{-ita}B(ds)e^{ita}) \tag{67}$$

Hence, by Propositions I.30 and II.22

$$E_{\Phi_t}(B) = \sum_j r_j \langle e^{-ita}be^{ita}e_j, e_j\rangle \tag{69}$$

Since the series is uniformly convergent (notice that $e^{-ita}be^{ita} \in \mathcal{B}_h$) and each term is infinitely differentiable with respect to t then

$$
\begin{aligned}
(d/dt)E_{\Phi_t}(B) &= \sum_{j\geq 1} r_j\ (d/dt)\langle e^{-ita}\,be^{ita}\ e_j, e_j\rangle \\
&= \sum_{j\geq 1} r_j \langle -iae^{-ita}\,be^{ita} + e^{-ita}\,b(ia)e^{ita}\ e_j, e_j\rangle \\
&= \sum_{j\geq 1} r_j \langle e^{-ita}(i[b,a])e^{ita}\ e_j, e_j\rangle \\
&= \mathrm{tr}(\hat{\Phi}_o e^{-ita}(i[b,a])e^{ita}) \\
&= \mathrm{tr}(e^{ita}\hat{\Phi}_o e^{-ita}(i[b,a])) \\
&= \Phi_t(i[b,a])
\end{aligned} \tag{70}
$$

From this equality and (65) we obtain the desired result. ∎

A special class of measurements associated with the type of transformations we have just seen is given in the following definition.

II.29 <u>DEFINITION:</u> A $\mathcal{A}$-measurement $B(.)$ is said to be unbiased with respect to the transformation $t \longmapsto \Phi_t$ given by (63.a) if

$$E_{\Phi_t}(B) = t \quad , \; t \in \mathbb{R} \qquad (71)$$

///

For any such measurement we have

II.30 <u>COROLLARY:</u> Using the notation above,

$$D_{\Phi_t}(B) D_{\Phi_t}(a) \geqslant \tfrac{1}{4} \qquad (72)$$

<u>PROOF.</u>

The result follows by replacing Eqn(71) in inequality (64). ∎

Now we shall further narrow the class of measurements associated with the type of transformations (63.a). Following Davies [2], we introduce the following definition

II.31 <u>DEFINITION:</u> An $\mathcal{A}$-measurement $B(.)$ is said to be covariant with respect to a one-parameter $\mathcal{A}$-unitary group $\{e^{ita} : t \in \mathbb{R}\}$ if for any $t \in \mathbb{R}$ and Borel set Δ

$$e^{-ita} B(\Delta) e^{ita} = B(\Delta - t) \qquad (73)$$

where

$$\Delta - t = \{s - t : s \in \Delta\} .$$

///

As an immediate consequence of this definition we have

II.32 <u>COROLLARY:</u>

(a) $\Phi_t(B(\Delta)) = \Phi_0(B(\Delta-t))$ (74)

for any Borel set Δ, $t\in\mathbb{R}$.

(b) $E_{\Phi_t}(B) = E_{\Phi_0}(B) - t$, $t\in\mathbb{R}$ (75)

(c) $D_{\Phi_t}(B) = D_{\Phi_0}(B)$, $t\in\mathbb{R}$ (76)

c.f.[2,66]. ///

Eqn(74) states that the shift $\Phi_0 \longmapsto \Phi_t$ of the initial state is reflected in the corresponding shift of the resulting probability distribution.

Eqn(75) states that any covariant measurement is unbiased up to an additive constant, while Eqn(76) states that any such measurement has constant variance.

In the next chapter we shall construct a special class of covariant measurements; these are based on the so called approximate position measurement introduced by Davies [2].

-.-

SUMMARY

The Davies and Lewis theory is naturally extended to cover the case of families of unbounded observables.The non reflexivity and not completness of $\mathcal{A}[u]$ forces us to take as a starting point the concept of an expectation.

An expectation is a normalized positive linear map on the observables; an instrument being the transposed of an expectation is a normalized positive linear map on the states. Any instrument is a bounded positive Radon mesure determining an $\langle\mathcal{A},W\rangle$-measure. However, any $\langle\mathcal{A},W\rangle$-measure is determined by many instruments.

An observable is measurable if there exist at least one instrument giving some information about this observable.

In general, two given instruments compose to a post instrument; again, this is a direct consequence of the non reflexivity of $\mathcal{A}[u]$.

For any two $\langle\mathcal{A},W\rangle$-measures a generalized version of the Heisenberg uncertainty relation holds.

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CHAPTER III

A CLASS OF INSTRUMENTS ON $S(\mathbb{R})$

In this chapter we consider the system $W[t]=S(\mathbb{R})$, with one degree of freedom. Our principle result is the explicit formula for a family of instrument which will measure the basic quantum mechanical operators with some degree of accuracy. We note that the basic formula was proposed by Davies [3] as a covariant approximate position instrument. What is new here is that we consider the family as labelled by the normalized elements of $S(\mathbb{R})$ and show that the result is an instrument in our sense, i.e., with reference to the algebra $\mathcal{A}$. We also show that the formula is valid for more operators than the position. Our principal result is this:

III.1 PROPOSITION: Let $a \in \mathcal{A}_h$ stand for any of the essentialy self-adjoint operators $Q, P=-iD$, or $H=P+V(Q)$, where $x \longmapsto V(x)$ is $C^{\infty}(\mathbb{R})$ and, together with all its derivatives, is bounded.

To each $f \in S(\mathbb{R})$ we associate the expectation

$$Z[a;f;\Delta](b) = \int_{\Delta} f_s(a)^* b f_s(a) ds \qquad (1)$$

where $f_s(a)$ is defined by the spectral calculus, with $f_s(x)=f(x-s)$:

$$f_s(a) = \int_{\mathbb{R}} f(x-s) E(dx) \qquad (2.a)$$

where

$$a = \int_{\mathbb{R}} xE(dx) \qquad (2.b)$$

by an abuse of notation we are not distinguishing $a \in \mathcal{A}_h$ form its closure a^{**}. The function f is required to be normalized:

$$\|f\|^2 = 1 \qquad (3)$$

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A few remarks about the physical interpretations of $Z[a;f;.]$ or, equivalently, its corresponding instrument $\mathcal{J}[a;f;.]=Z[a;f;.]^t$, are pertinent here.

Were it possible to build a perfect instrument, and if is not, it would be the simple generalization of the discrete formula, viz,

$$Z_\infty[a;\Delta](b) = \int_\Delta E(s)bE(s)ds \qquad (4)$$

One reason why Z_∞ is not an instrument is that it «chops off the incoming wave functions too sharply» at the boundaries of Δ. As it is required that «$S(\mathbb{R})$ in, $S(\mathbb{R})$ out», we must smooth Z out. Hence regularizing a with f. More precisely:

III.2 <u>COROLLARY:</u> The $(\mathcal{A},S)$-measure $M(\mathcal{J};\Delta)$ corresponding to the instrument $\mathcal{J}=\mathcal{J}[a;f;.]$ is

$$M(\mathcal{J};\Delta) = \int_{\mathbb{R}} (F*\chi_\Delta)(x)E(dx)$$
$$= (F*\chi_\Delta)(a) \qquad (5)$$

with $F(x)=|f(x)|^2$, and convolution is meant. ///

This formula was given by Davies; that it defines an $(\mathcal{A},S)$-measure is a consequence of our general theory, Proposition II.12, once we show Z to be an expectation.

One measure of the goodness of an instrument, probably not a useful measure, is the difference between the instrument $(\mathcal{A},S)$-measure M and the $\mathcal{A}$-measure in question, here E.

If we consider, in the $\mathbb{B}(\mathcal{H})$ norm

$$\| (F*\chi_{\Delta})(a)-E(\Delta) \| \quad ,$$

we see that this vanishes for $F=\delta$ (delta distribution). As $F\in S(\mathbb{R})$ and $S(\mathbb{R})$ is dense in $S'(\mathbb{R})$, choosing F close to δ in some way makes $\mathcal{S}$ a good instrument. Similarly, choosing F close to he constant function makes $\mathcal{S}$ into a poor instrument.

Before we go into the proof of proposition III.1 we shall introduce the operators P,Q and H for which instruments are to be constructed.

THE SCHWARTZ SPACE $S=S(\mathbb{R})$ AND THE OPERATORS Q,P AND H

III.3 DEFINITION: The Schwartz space S is the set of all infinitely differentiable complex valued functions $f(x)$ on $\mathbb{R}$ for which

$$\|\varphi\|_{m,n;\infty}= \sup_{x\in\mathbb{R}} |x^m D^n \varphi(x)| < \infty \tag{6}$$

where $m,n\in\mathbb{N}_0$ and $D^n=(d^n/dx^n)$

It is well known that S endowed with the natural topology given by the seminorms $\|.\|_{m,n;\infty}$ is a Frechet space (c.f.[26]). This topology can also be generated by each one of the following set of seminorms [26]

$$\|\varphi\|_{m,n;2}= \|x^m D^n\varphi\|_2 \tag{7}$$

$$\|\varphi\|_{k;\infty} = \max \{\|\varphi\|_{m,n;\infty}: m \leqslant k,\ n \leqslant k-m\} \tag{8}$$

$$\|\varphi\|_{k;2} = \max\{\|\varphi\|_{m,n;2} : m \leqslant k,\ n \leqslant k-m\} \qquad (9)$$

where $m,n,k \in \mathbb{N}_0$ and $\|\cdot\|$ is the usual $L^2(\mathbb{R})$-norm. ///

The operators P,Q and H of proposition III.1 have S as a natural domain of definition.As we shall see there exist a close relationship between these operators and the Schwartz topology on S introduced above.

III.4 <u>SCHRÖDINGER REPRESENTATION OF THE CANONICAL COMMUTATION RELATIONS FOR ONE DEGREE OF FREEDOM:</u> Let $\mathcal{U}$ be the free non-commutative algebra of all polynomials in the hermitian generators $\{q,p\}$ modulo the two-sided $*$-ideal generated by the relations

$$[q,p] = qp-pq = iI$$
$$[q,q] = [p,p] = 0 \qquad (10)$$

We define a $*$-representation π on $L^2(\mathbb{R})$ by defining

$$(\pi(q)\varphi)(x) = x\varphi(x)$$
$$(\pi(p)\varphi)(x) = -i(d\varphi/dx)(x) \qquad (11)$$

The domain of π, $\mathcal{D}(\pi)$ is taken to be the Schwartz space S. It can be shown that π is an irreducible self-adjoint $*$-representation of $\mathcal{U}$ and that the induced topology on $\mathcal{D}(\pi)=S$ is equivalent to the Schwartz space topology given by (6); c.f.[50,51].

Set now $H=(\frac{1}{2})(p^2+q^2+I)$, and let $\mathcal{U}_1$ be the $*$-subalgebra of $\mathcal{U}$ consisting of all polynomials in H . Then, $\pi(H^k)=2^{-k}(-\Delta+x^2+1)^k\restriction S$ is essentially self-adjoint and a function $\varphi \in L^2(\mathbb{R})$ is in S if and only if $\varphi \in D(\overline{\pi(H^k)})$ (the domain of the closure of $\pi(H^k)$), $k \in \mathbb{N}_0$, that is

$$S = \bigcap_{k \in \mathbb{N}_0} D(\overline{\pi(H^k)}) \qquad (12)$$

Furthermore, the topology on S defined by the seminorms

$$\|\varphi\|_k = \|\pi(H^k)\varphi\|_2 \quad , \; k \in \mathbb{N}_0 \qquad (13)$$

is equivalent to the Schwartz space topology on S.

On the other hand, since the eigenvalues of $\pi(H)$ are of the form $\lambda_n = n+1$; $n \in \mathbb{N}_0$, then $\lim_n \lambda_n = +\infty$ and so by the well-known theory in a Hilbert space $(\pi(H^2)+I)^{-1}$ is completely continuous. Hence S is a wave function space, for a system with one degree of freedom, in the sense of Definition I.6.

Setting $P = \overline{\pi(p)}$ and $Q = \overline{\pi(q)}$ we find from Eqns(11), for any $\varphi \in S$

$$Q\varphi \in S, \; P\varphi \in S$$

$$Q^*\varphi = Q\varphi \, , \quad P^*\varphi = P\varphi$$

$$[Q,P]\varphi = QP\varphi - PQ\varphi = i\varphi$$

$$Q^*\varphi = Q\varphi \quad , \quad P^*\varphi = P\varphi \qquad (14)$$

The self-adjoint operators Q and P are known as the representation on $L^2(\mathbb{R})$ of the position and momentum observables of a spinless particle with 1-degree of freedom. It is clear from (14) that the algebra generated by Pand Q is contained in $\mathcal{A}$.

Now we turn our attention to the class of Hamiltonian operators described in proposition III.1.

III.5 <u>DEFINITION:</u> An S-Hamiltonian operator is a self-adjoint operator H in $\mathcal{H} = L^2(\mathbb{R})$ of the form

$$H = P^2 + U(Q) \tag{15}$$

so that $\{e^{itH}: t\in\mathbb{R}\}$ is a one-parameter $\mathcal{A}$-unitary group. ///

It follows from this definition and definition II.26 that any S-Hamiltonian operator satisfies

$$\text{(i)}\ H[S]\subset S \quad , \text{ i.e. } H\in\mathcal{B}_h \tag{16}$$

$$\text{(ii)}\ e^{itH}[S]\subset S \ ,\ t\in\mathbb{R}. \tag{17}$$

As we shall see, operators of the above class are abundant and an important number of them are, in a sense, closely connected to the topological structure of S.

Set $H_0=P^2$, and let $U(x)$ be a bounded C^∞-function on R with bounded derivatives (the bounds depending on the derivatives), and U be the operator of multiplication by $U(x)$, defined on all functions $\varphi\in L^2(R)$ for which this product is again in $L^2(R)$.

It is straightforward to show that U satisfies the Kato condition:

$D(H_0)\subset D(U)$, and there exist non-negative constants $a<1$, $b<\infty$ such that

$$\|U\varphi\| \leq a\|H_0\varphi\| + b\|\varphi\| \quad , \varphi\in D(H_0) \tag{18}$$

Hence , H has domain $D(H_0)$ and is self-adjoint ([34],[38:Thm (5.5)]). Furthermore, since the norms $\|H\varphi\|+\|\varphi\|$ and $\|H_0\varphi\|+\|\varphi\|$ are equivalent on $D(H_0)$ and $H_0=\overline{H_0\restriction S}$, then $H=\overline{H\restriction S}$.

Now for any index $n\in\mathbb{N}_0$,let D_n be the linear subset of $L^2(\mathbb{R})$ defined by

$$D_n=\bigcap_{\substack{k\leq n\\ m\leq n-k}} D(Q^k H^m) \tag{19.a}$$

and equipped with the norm

$$\|\varphi\|_{\underline{n};2}=\sup_{\substack{k\leq n\\ m\leq n-k}} \|Q^k H^m\varphi\|_2 \quad , \varphi\in D_n \tag{20.a}$$

It follows from the above discussion that for any $n \in \mathbb{N}_0$

$$D_n = \bigcap_{\substack{k \leq n \\ m \leq n-k}} D(Q^k H_0^m) \qquad (19.b)$$

and the system of norms $\{\|\cdot\|_{\underline{n};2} : n \in \mathbb{N}_0\}$ is equivalent to the system of norms $\{\|\cdot\|'_{\underline{n};2} : n \in \mathbb{N}_0\}$ defined by

$$\|\varphi\|'_{\underline{n};2} = \sup_{\substack{k \leq n \\ m \leq n-k}} \|Q^k H_0^m \varphi\|_2 \quad , n \in \mathbb{N}_0 \qquad (20.b)$$

Furthermore

$$S(\mathbb{R}) = \bigcap_{n \in \mathbb{N}_0} D_n \qquad (21)$$

and the family of norms $\|\cdot\|'_{\underline{n};2}$, or equivalently $\|\cdot\|_{\underline{n};2}$, generates the Schwartz topology on S.

That H is an S-Hamiltonian operator follows from the following theorem of Hunziker [31:Thm 2].

III.6 <u>THEOREM:</u> If $V(x)$ is a C^∞- function with bounded derivatives, then S is invariant under $H = P^2 + V(Q)$ and the unitary group $\{e^{-itH} : t \in R\}$. Moreover, the mapping $(f,t) \mapsto e^{-itH} f$ of $S \times R$ onto S is continuous (in the sense of the Schwartz topology on S). ///

III.7 <u>REMARK:</u>

(a) It is an open question whether any S-Hamiltonian operator generates the Schwartz topology in the manner described by equations (19-21).

(b) In what follows we only consider S-Hamiltonian operators H satisfying the conditions of Theorem III.6.

FUNCTIONAL CALCULUS FOR Q,P AND H

First, we shall prove a number of useful inequalities involving the operators Q,P and H.

III.8 LEMMA:

(i) $$Q^m P^n e^{iaQ} = e^{iaQ} \sum_{k\leq n} \binom{n}{k} a^{n-k} Q^m P^k \tag{22.a}$$

(ii) $$Q^m P^n e^{ibP} = e^{ibP} \sum_{k\leq m} \binom{m}{k} b^{m-k} Q^k P^n \tag{22.b}$$

(iii) $$Q^n e^{-itH} = e^{-itH} Q^n + i \int_0^t e^{-i(t-s)H} [H,Q^n]\, e^{-isH}\, ds \tag{22.c}$$

with

$$[H,Q^n] = 2in\, PQ^{n-1} + n(n-1)Q^{n-2} \tag{22.d}$$

(iv) $$\| e^{iaQ}\varphi \|_{m,n;2} \leqslant \sum_{k\leq n} \binom{n}{k} |a|^{n-k} \|\varphi\|_{m,k;2} \tag{22.e}$$

$$\leqslant (1+|a|)^n \|\varphi\|_{m+n;2} \tag{22.f}$$

and so

$$\| e^{iaQ}\varphi \|_{r;2} \leqslant (1+|a|)^r \|\varphi\|_{r;2} \tag{22.g}$$

(v) $$\| e^{ibP}\varphi \|_{m,n;2} \leqslant \sum_{k=0}^{m} \binom{m}{k} |b|^{m-k} \|\varphi\|_{k,n;2} \tag{22.h}$$

$$\leqslant (1+|b|)^m \|\varphi\|_{m+n;2} \tag{22.i}$$

and so

$$\| e^{ibP}\varphi \|_{r;2} \leqslant (1+|b|)^r \|\varphi\|_{r;2} \tag{22.j}$$

(vi) $$\| e^{-itH}\varphi \|_{\underline{n};2} \leqslant c_n (1+|t|)^n \|\varphi\|_{\underline{n};2} \tag{22.k}$$

PROOF.

For a proof of (iii) and (vi) see [31].Of the remaining statements we only prove (i) and (iv),for (ii) and (v) are proved in a similar way.

Now, from the Weyl relation

$$e^{ibP}\, e^{iaQ} = e^{iab}\, e^{iaQ}\, e^{ibP} \; ,$$

by differentiation at the point $b=0$ we obtain

$$Pe^{iaQ} = e^{iaQ}\,(P+a)\varphi \; . \tag{23.a}$$

Using this expresion inductively, for any $n,m\in \mathbb{N}_0$ we get

$$Q^m P^n e^{iaQ} = e^{iaQ} \sum_{k=0}^{n} \binom{n}{k} a^{n-k} Q^m P^k \varphi \; .$$

This shows (i). Furthermore

$$\begin{aligned} \|Q^m P^n e^{iaQ}\varphi\|_2 &\leq \sum_{k=0}^{n} \binom{n}{k} |a|^{n-k} \|Q^m P^k \varphi\|_2 \\ &= \sum_{k=0}^{n} \binom{n}{k} |a|^{n-k} \|\varphi\|_{m,k;2} \\ &\leq \|\varphi\|_{m+n;2} \sum_{k=0}^{n} \binom{n}{k} |a|^{n-k} \\ &= \|\varphi\|_{m+n;2}\, (1+|a|)^n \; , \end{aligned} \tag{23.b}$$

Hence

$$\|e^{iaQ}\varphi\|_{r;2} \leq (1+|a|)^r\, \|\varphi\|_{r;2} \qquad , r\in\mathbb{N}_0 \tag{23.c}$$

The inequalities (23.b) and (23.c) show that (i) and (iv) are true. ∎

Our next lemma provides us with a functional calculus for Q, P and H. This construction is essential to the extent that it ensures that the expectations of proposition III.1 are well defined.

III.9 <u>LEMMA:</u> Let g be a multiplier of S, ie. $g\in C^\infty(\mathbb{R})$ and g and each of its derivatives are polynomially bounded, then

(i) $g(Q)\,[S]\subset S$ (24.a)

(ii) $P\, g(Q)\varphi - g(Q)\, P\varphi = -ig'(Q)\varphi \quad , \varphi\in S$ (24.b)

(iii) $g(P)\,[S]\subset S$ (25.a)

(iv) $g(P)\, Q\varphi - Q\, g(P)\varphi = -ig'(P) \quad , \varphi\in S$ (25.b)

Let g be an element of $\mathcal{E}$, the set of Fourier transform of functions $f:\mathbb{R} \longrightarrow \mathbb{C}$ satisfying

$$f(t)\ (1+|t|)^n \in L^1(\mathbb{R}) \quad , n\in\mathbb{N}_0 \qquad (26.a)$$

then

$$\text{(v)} \qquad g(H)\ [S] \subset S\ . \qquad (26.b)$$

PROOF.

(i) to (iv) have been proved by Schmudgen [48].

To see (v), let $\varphi\in S$ and $g=\tilde{f}$, for some f satisfying (26.a). Then by Fubini's theorem for any $\psi\in L^2(R)$

$$\langle g(H)\varphi , \psi \rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t)\ \langle e^{-itH}\varphi , \psi\rangle\ dt\ . \qquad (27)$$

Using lemma III.8.(vi), for any $n\in\mathbb{N}_0$

$$\begin{aligned} \| g(H)\varphi \|_{\underline{n};2} &\leqslant \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |f(t)|\ \| e^{-itH}\varphi \|_{\underline{n};2}\ dt \\ &\leqslant \int_{\mathbb{R}} |f(t)|\ c_n\ (1+|t|)^n\ \|\varphi\|_{\underline{n};2}\ dt \\ &= c_n\ \|\varphi\|_{\underline{n};2} \int_{\mathbb{R}} |f(t)|\ (1+|t|)^n\ dt < \infty\ , \end{aligned} \qquad (28)$$

From which the result follows; since by (21) the norms $\{\|\ .\ \|_{\underline{n};2} : n\in\mathbb{N}_0\}$ generate the Schwartz topology. ■

As a direct consequence of this lemma we have that for any $g\in S$ and $a=Q,P,H$, $g(a)$ belongs to $\mathcal{A}$. If in addition g is real valued then $g(a)\in\mathcal{A}_h$ and is self-adjoint.

III.10 <u>COROLLARY:</u> For any $m,n\in\mathbb{N}_0$, $\varphi\in S$

$$P^m Q^n \varphi = \sum_{k=0}^{m} \binom{m}{k}\binom{n}{k}\ k!\ (-i)^k\ Q^{n-k} P^{m-k} \varphi \qquad (29)$$

where by convention $\binom{n}{k}=0$ if $n-k<0$.

Hence

$$\|Q^n\varphi\|_{a,b;2} \leqslant b!\ 2^{b+n}\ \|\varphi\|_{a+b+n;2} \tag{30.a}$$

and so

$$\|Q^n\varphi\|_{N;2} \leqslant N!\ 2^{N+n}\ \|\varphi\|_{N+n;2} \tag{30.b}$$

PROOF.

Equation (29) follows from equations (24.b), (25.b)(with $g(x)=x^{\ell}$), and by induction on m and n: eg. for instance fix n; for $m=1$ eqn (29) is just eqn (24.b) with $g(x)=x^n$. Assume now that (29) is true for $m=M$. Then, using (29) and (24.b) we get

$$\begin{aligned} P^{M+1}Q^n\varphi &= \sum_{k=0}^{M}\binom{M}{k}\binom{n}{k}\ k!\ (-i)^k\,[Q^{n-k}P - i(n-k)Q^{n-k-1}]P^{M-k}\varphi \\ &= \sum_{k=0}^{M}\binom{M}{k}\frac{n!}{n-k!}\,(-i)^k\,Q^{n-k}\,P^{M-k+1}\varphi + \sum_{k=0}^{M}\binom{M}{k}\frac{n!}{(n-k-1)!}\,(-i)^k\,Q^{n-k-1}P^{M-k}\varphi \end{aligned}$$

Changing variables in the second sum, $k+1 \longmapsto k$, and then expanding both terms conveniently

$$\begin{aligned} &= \binom{M}{0}\,Q^nP^{M+1}\varphi + \sum_{k=1}^{M}\left[\binom{M}{k}+\binom{M}{k-1}\right]\frac{n!}{(n-k)!}\,(-i)^k\,Q^{n-k}P^{M+1-k}\varphi + \\ &\quad + \binom{M}{M}\frac{n!}{(n-(M+1))!}\,(-i)^{M+1}\,Q^{n-(M+1)}P^{0}\varphi \\ &= \binom{M+1}{0}\,Q^nP^{M+1}\varphi + \sum_{k=1}^{M}\binom{M+1}{k}\binom{n}{k}\ k!\ (-i)^k\,Q^{n-k}P^{M+1-k}\varphi + \\ &\quad + \binom{M+1}{M+1}\binom{n}{M+1}\,(m+1)!\ (-i)^{M+1}\,Q^{n-(M+1)}P^{0}\varphi \\ &= \sum_{k=0}^{M+1}\binom{M+1}{k}\binom{n}{k}\ k!\ (-i)^k\,Q^{n-k}\,P^{M+1-k}\varphi\ . \end{aligned}$$

This shows that (29) is true for $m=M+1$. The induction on n is carried out in the same way, so we are done.

To show (30.a) we notice that by (29) we have

$$Q^aP^bQ^n\varphi = \sum_{k=0}^{b}\binom{b}{k}\binom{n}{k}\ k!\ (-i)^k\ Q^{a+n-k}\,P^{b-k}\varphi\ ,$$

hence

$$\begin{aligned} \|Q^n\varphi\|_{a,b;2} &\leqslant b!\ \sum_{k=0}^{b}\binom{b}{k}\binom{n}{k}\ \|\varphi\|_{a+n-k,b-k;2} \\ &\leqslant b!\ \|\varphi\|_{a+b+n;2}\sum_{k=0}^{b}\binom{b}{k}\binom{n}{k} \end{aligned}$$

Using now the inequality

$$\sum_{k=0}^{b}\binom{b}{k}\binom{n}{k} \leqslant \left[\sum_{k=0}^{b}\binom{b}{k}\right]\left[\sum_{k=0}^{b}\binom{n}{k}\right] \leqslant 2^{b+n}$$

yields (30.a).

The proof of (30.b) is trivial. ∎

PREPARATORY LEMMAS

The following sequence of lemmas will provide us with useful estimates for the inequalities we are going to find in the sequel.

We shall start by showing that the Fourier transform of the characteristic function of a Borel set is a tempered distribution.

III.11 LEMMA: For any Borel set $\Delta\in Bor(\mathbb{R})$, the Fourier transform of χ_Δ, $\sqrt{2\pi}\,\delta_\Delta$, is a tempered distribution.

Consequently, for each Borel set there exist an index M and a constant C, both independent of Δ, such that for every $f\in S$

$$|\delta_\Delta(f)| \leqslant C\, \|f\|_{M;\infty} \tag{31}$$

PROOF.

It is sufficient to show that $\chi_\Delta\in S'$.

For all $f\in S$, multiplying and dividing by $(1+|t|^2)$ gives

$$|\chi_\Delta(f)| = \left|\int_\Delta f(t)\ dt\ \right|$$

$$\leqslant \sup_{t\in\mathbb{R}} |(1+t^2)\, f(t)| \int_{\Delta} (1+t^2)^{-1}\, dt$$
$$\leqslant 2\pi\, \|f\|_{2;\infty}.$$

To show that the constant C is independent of Δ we notice

$$\begin{aligned} \delta_{\Delta}(f) &= \int_{\mathbb{R}} \delta_{\Delta}(s)\, f(s)\, ds \\ &= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \chi_{\Delta}(t)\, \frac{e^{-ist}}{2\pi}\, dt\Big)\, f(s)\, ds \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \chi_{\Delta}(t)\, \tilde{f}(t)\, dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{\Delta} \tilde{f}(t)\, dt \quad . \end{aligned}$$

Hence

$$\begin{aligned} |\delta_{\Delta}(f)| &\leqslant \sup_{t\in\mathbb{R}} |(1+t^2)\, \tilde{f}(t)| \int_{\Delta}(1+t^2)^{-1}\, dt \\ &\leqslant 2\pi\, \|\tilde{f}\|_{2;\infty} \\ &\leqslant C\, \|f\|_{M;\infty}. \end{aligned}$$

In the last inequality we have used the continuity of the Fourier transform in the Schwartz topology. ∎

III.12 <u>LEMMA:</u> Let $f\in S$ be the function labelling the instrument and $g\in S$ be the function whose Fourier transform is f ; ie. $g(t)=\mathcal{F}[f](-t)$.

Let $a\in\mathcal{A}_h$ stand for any of the essentially self-adjoint operators Q, P or H. Let $b\in\mathcal{A}$ and write

$$b_t = e^{ita}\, b\, e^{-ita} \tag{32}$$

so that $b_t\in\mathcal{A}$ for each $t\in R$. Let $\varphi,\psi\in S$ be arbitrary and consider the function

$$s \longmapsto h(s) = \bar{g}(s+t)\, \langle e^{isa}\, b_t\varphi, \psi\rangle_{v,w;2} \tag{33}$$

where

$$\langle e^{isa}\, b_t\varphi, \psi\rangle_{v,w;2} = \begin{cases} \langle Q^v P^w e^{isa}\, b_t\varphi, \psi\rangle & \text{if } a=Q,P \\ \langle Q^v H^w e^{isa}\, b_t\varphi, \psi\rangle & \text{if } a=H \end{cases} \tag{34}$$

and so, eg.

$$|\langle \varphi,\psi\rangle_{v,w}| \leqslant \|\psi\|\ \|\varphi\|_{\widetilde{v+w};2} \tag{35}$$

where

$$\tilde{k} = \begin{cases} k & \text{if } a=Q,P \\ \underline{k} & \text{if } a=H \end{cases}$$

Then $h \in S$, with

$$\|h\|_{M;\infty} \leqslant \|\psi\| \cdot \begin{Bmatrix} (v+w)!\,2^{2(v+w+M)} & a=Q \\ 2^{v+w+M} & a=P \\ C_{v,w}\cdot 2^{v+w+M} & a=H \end{Bmatrix} \cdot \|b_t\varphi\|_{\widetilde{v+w+M};2}\,(1+|t|)^{v+w+M}\,\|g\|_{v+w+M;\infty} \tag{36.a}$$

or

$$\|h\|_{M;\infty} \leqslant C_{v,w,M}\|\psi\|\ (1+|t|)^{v+w+M}\|b_t\varphi\|_{\widetilde{v+w+M};2}\|g\|_{v+w+M;\infty} \tag{36.b}$$

PROOF.

Let us abbreviate d/ds to D_s. The map $s \longmapsto \langle e^{isa}\ b_t\varphi\ ,\psi\rangle_{v,w}$ is $C^\infty(R)$ and, for any $k\in \mathbb{N}_0$

$$\begin{aligned} D_s^k\ \langle e^{isa}\ b_t\varphi\ ,\psi\rangle_{v,w} &= (i\ )^k\langle a^k e^{isa}\ b_t\varphi\ ,\psi\rangle_{v,w} \\ &= (i\ )^k\langle e^{isa}\ a^k b_t\varphi\ ,\psi\rangle_{v,w} \end{aligned} \tag{37}$$

Using lemma III.8 and inequality (35) we obtain the following estimates

$$\left|D_s^k\langle e^{isa}\ b_t\varphi\ ,\psi\rangle_{v,w}\right| \leqslant \|\psi\|\ (1+|s|)^{v+w}\cdot\begin{Bmatrix} 1 & \text{if } a=Q,P \\ C_{v,w} & \text{if } a=H \end{Bmatrix}\cdot\|a^k b_t\varphi\|_{\widetilde{v+w};2} \tag{38}$$

By corollary III.10 and the definition of the norms $\|\cdot\|_{n;2}$, $\|\cdot\|_{\underline{n};2}$ we finally get

$$\left|D_s^k\langle e^{isa}\ b_t\varphi\ ,\psi\rangle_{v,w}\right| \leqslant \|\psi\|(1+|s|)^{v+w}\cdot\begin{Bmatrix} (v+w)!\,2^{v+w+k} & a=Q \\ 1 & a=P \\ C_{v,w} & a=H \end{Bmatrix}\cdot\|b_t\varphi\|_{\widetilde{v+w+k};2} \tag{39}$$

Now

$$D_s^n h(s)=\sum_{r=0}^{n}\binom{n}{r} D_s^r(\bar{g}(s+t))\; D_s^{n-r}\langle e^{isa}\; b_t\varphi ,\psi\rangle_{v,w} \qquad (40)$$

so that

$$\|h\|_{m,n;\infty}=\sup\left\{\left|s^m D_s^n h(s)\right| : s\in\mathbb{R}\right\}$$

$$\leqslant \sum_{r=0}^{n}\binom{n}{r} \sup_{s\in\mathbb{R}} \left|s^m D_s^r(\bar{g}(s+t))\; D_s^{n-r}\langle e^{isa}\; b_t\varphi ,\psi\rangle_{v,w}\right|$$

$$\leqslant \sum_{r=0}^{n}\binom{n}{r} \|\psi\| \sup_{s\in\mathbb{R}}\left|(1+|s|)^{v+w} \cdot \begin{cases}(v+w)!\,2^{v+w+n-r} & a=Q\\ 1 & a=P\\ C_{v,w} & a=H\end{cases} \cdot \|b_t\varphi\|_{\widetilde{v+w+n-r;2}} \cdot s^m D_s^r(\bar{g}(s+t))\right| \qquad (41)$$

$$\leqslant \|\psi\| \cdot \begin{cases}(v+w)!\,2^{v+w+n} & a=Q\\ 1 & a=P\\ C_{v,w} & a=H\end{cases} \cdot \|b_t\varphi\|_{\widetilde{v+w+n;2}} \cdot \sum_{r=0}^{n}\binom{n}{r} \cdot \sup_{s\in\mathbb{R}}\left|s^m(1+|s|)^{v+w}\; D_s^r\bar{g}(s+t)\right| \qquad (42)$$

Set $\alpha=v+w$, then

$$\sup_{s\in\mathbb{R}}\left|s^m(1+|s|)^{\alpha} D_s^r\bar{g}(s+t)\right| \leqslant \sum_{k=0}^{\alpha}\binom{\alpha}{k} \sup_{s\in\mathbb{R}}\left|s^{m+\alpha-k} D_s^r\bar{g}(s+t)\right|$$

$$=\sum_{k=0}^{\alpha}\binom{\alpha}{k}\|\bar{g}_{-t}\|_{m+\alpha-k,r;\infty}$$

$$\leqslant \|\bar{g}_{-t}\|_{m+\alpha+r;\infty}\,2^{\alpha}$$

$$\leqslant \|\bar{g}_{-t}\|_{v+w+m+n;\infty}\; 2^{v+w}$$

$$\leqslant (1+|t|)^{v+w+m+n}\|g\|_{v+w+m+n;\infty}\,2^{v+w} \qquad (43)$$

For the last inequality we have used the fact that

$$\|\bar{g}_{-t}\|_{M;\infty} \leqslant (1+|t|)^{M}\; \|g\|_{M;\infty}\;, \; M\in\mathbb{N}_0 \qquad (44)$$

Using (43) we now find an upper bound for the last factor, (#), of expression (42)

$$(\#) \leqslant \sum_{r=0}^{n}\binom{n}{r}\; 2^{v+w}\; (1+|t|)^{v+w+m+n}\|g\|_{v+w+m+n;\infty}$$

$$\leqslant 2^{v+w+n}\; (1+|t|)^{v+w+m+n}\|g\|_{v+w+m+n;\infty} \qquad (45)$$

Hence

$$\|h\|_{m,n;\infty} \leqslant \|\psi\| \cdot \begin{Bmatrix} (v+w)!\,2^{2(v+w+n)} & a=Q \\ 2^{v+w+n} & a=P \\ C_{v,w}\,2^{v+w+n} & a=H \end{Bmatrix} \cdot \|b_t\varphi\|_{\widetilde{v+w+n};2} (1+|t|)^{v+w+m+n} \|g\|_{v+w+m+n;\infty} \tag{46}$$

and so

$$\|h\|_{M;\infty} \leqslant \|\psi\| \cdot \begin{Bmatrix} (v+w)!\,2^{2(v+w+M)} & a=Q \\ 2^{v+w+M} & a=P \\ C_{v,w}\,2^{v+w+M} & a=H \end{Bmatrix} \cdot \|b_t\varphi\|_{\widetilde{v+w+M};2} (1+|t|)^{v+w+M} \|g\|_{v+w+M;\infty} \tag{47}$$

and we are done. ∎

III.13 <u>COROLLARY:</u> Using the notation of the above lemma, consider the function

$$s \longmapsto h(s)=\bar{g}(s+t)\ \langle e^{isa}\ b_t\varphi\,,\psi\rangle_{N;2}\ , \tag{48}$$

where

$$\langle \varphi,\psi\rangle_{N;2} = \underset{\substack{\text{in abs.}\\ \text{val.}}}{\max}\ \{\ \langle \varphi,\psi\rangle_{m,n;2}\ : m\leqslant N,\ n\leqslant N-m\ \} \tag{49}$$

so that

$$|\langle \varphi,\psi\rangle_{N;2}| \leqslant \|\psi\|\ \|\varphi\|_{\tilde{N};2}\ , \tag{50}$$

Then $h\in S$ and

$$\|h\|_{M;\infty} \leqslant \|\psi\| \cdot \begin{Bmatrix} N!\,2^{2(N+M)} & a=Q \\ 2^{N+M} & a=P \\ C_{N,M}\,2^{N+M} & a=H \end{Bmatrix} \cdot \|b_t\varphi\|_{\widetilde{N+M};2} (1+|t|)^{N+M} \|g\|_{N+M;\infty} \tag{51.a}$$

or

$$\|h\|_{M;\infty} \leqslant \|\psi\|\ C_{N,M}\ (1+|t|)^{N+M} \|b_t\varphi\|_{\widetilde{N+M};2} \|g\|_{N+M;\infty} \tag{51.b}$$

///

It will be necessary to find an estimate in t for $\|b_t\varphi\|$

III.14 <u>LEMMA:</u> For each b and N+M there exists an index P and a positive constant $C_{N,M}$ such that for all $t\in\mathbb{R}$ and all $\varphi\in S$

$$\| b_t\varphi\|_{\widetilde{N+M};2} \leqslant C_{N,M}\ (1+|t|)^{N+M+P}\ \|\varphi\|_{\tilde{P};2} \tag{52}$$

PROOF.

We use inequalities (22.g),(22.j) and (22.k) to get

$$\begin{aligned}\| b_t\varphi\|_{\widetilde{N+M};2} &= \| e^{ita}\ be^{-ita}\varphi\|_{\widetilde{N+M};2}\\ &\leqslant (1+|t|)^{N+M}\ \| be^{-ita}\varphi\|_{\widetilde{N+M};2}\cdot\begin{cases}1 & a=Q,P\\ C_{N+M} & a=H\end{cases}\end{aligned} \tag{53}$$

The continuity of b gives

$$\| b\psi\|_{\widetilde{N+M};2} \leqslant \tilde{C}_{N+M}\ \|\psi\|_{\tilde{P};2} \tag{54}$$

where P depends on N and M (in fact, it depends on N+M).

Puting $\psi=e^{-ita}\varphi$ in (54) and using again (22.g),(22.j) and (22.k) we obtain

$$\| be^{-ita}\varphi\|_{\widetilde{N+M};2} \leqslant \tilde{C}_{N+M}\cdot\begin{Bmatrix}1 & a=Q,P\\ C_P & a=H\end{Bmatrix}\cdot(1+|t|)^P\ \|\varphi\|_{\tilde{P};2} \tag{55}$$

Replacing this inequality in (53) yields (52). ∎

Our last lemma gives us some fairly simple and useful estimates for some of the integrals we shall find in the sequel.

III.15 LEMMA: Let g belong to S , then

(i) $$\int_{\mathbb{R}}(1+|t|)^N\ |g(t)|\ dt \leqslant \pi\, 2^{N+2}\|g\|_{N+2;\infty}\quad ,N\in\mathbb{N}_0 \tag{56}$$

(ii) $$\int_{\mathbb{R}} |t|^k\ |g(t)|^2\ dt\ \leqslant\ \|g\|^2_{2;2}\qquad ,k=0,1,2 \tag{57}$$

PROOF.

(i) $$\int_{\mathbb{R}}(1+|t|)^N\ |g(t)|\ dt \leqslant \int_{\mathbb{R}}(1+t^2)^{-1}\ dt\ .\sup_{t\in\mathbb{R}}\left|(1+|t|)^{N+2}g(t)\right|$$

$$\leqslant \pi \sum_{k=0}^{N+2} \binom{N+2}{k} \|g\|_{N+2-k,0;\infty}$$
$$\leqslant \pi \max\{\|g\|_{N+2-k,0;\infty} : 0 \leqslant k \leqslant N+2\} \sum_{k=0}^{N+2} \binom{N+2}{k}$$
$$\leqslant \pi\, 2^{N+2} \|g\|_{N+2;\infty} .$$

(ii) For k=0,2 the result is clear. For k=1 we have

$$\int_{\mathbb{R}} |t|\ |g(t)|^2\ dt \leqslant (\tfrac{1}{2}) \int_{\mathbb{R}} (1+t^2)|g(t)|^2\ dt$$
$$= (\tfrac{1}{2})(\|g\|^2_{0,0;2} + \|g\|^2_{2,0;2})$$
$$\leqslant \|g\|^2_{2;2} .$$

This completes the proof of the lemma. ∎

PROOF OF THE MAIN RESULT

PROOF OF PROPOSITION III.1 :

We must show that

$$\|Z(\Delta)b\,\varphi\|_{N;2} = \sup\{|\langle Z(\Delta)[b]\varphi, \psi\rangle_{N;2}| : \psi \in S,\ \|\psi\| = 1\}$$

is finite for all Δ, b, ψ.

Using Fourier transform

$$f(x-s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-it(x-s)}\ g(t)\ dt ,$$

hence

$$\langle f_w(a)\varphi, \psi\rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-it(x-w)}\ g(t)\ dt\ \langle E(dx)\varphi, \psi\rangle$$
$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(t)\ e^{itw}\ \langle e^{-ita}\varphi, \psi\rangle\ dt,$$

Similarly

$$\langle f_w(a)^*\varphi, \psi\rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \bar{g}(s)\ e^{-isw}\ \langle e^{isa}\varphi, \psi\rangle\ ds .$$

Thus

$$\int_{\Delta} \langle f_w(a)^* b f_w(a)\varphi, \psi\rangle\ dw =$$
$$= \int_{\Delta} \int_{\mathbb{R}^2} \bar{g}(s)\ g(t)\ \frac{e^{-i(s-t)w}}{2\pi} \langle e^{isa}\ b e^{-ita}\varphi, \psi\rangle\ dt\ ds\ dw$$

Since $\bar{g}(s)\, g(t)\, e^{-i(s-t)w} \langle e^{isa}\, be^{-ita}\varphi, \psi\rangle$ is measurable then by Fubini's theorem we can interchange the order of integration. Thus, changing variables $s-t \longmapsto s$ we get

$$\langle Z(\Delta)[b]\varphi, \psi\rangle = \int_{\mathbb{R}^2} g(t)\, \bar{g}(s+t)\, \langle e^{isa}\, b_t\varphi, \psi\rangle \left(\int_\Delta \frac{e^{-isw}}{2\pi}\, dw\right) ds\, dt$$
$$= \int_{\mathbb{R}} g(t) \int_{\mathbb{R}} \bar{g}(s+t)\, \langle e^{isa}\, b_t\varphi, \psi\rangle\, \delta_\Delta(s)\, ds\, dt$$

where $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (58)

$$\delta_\Delta(s) = \int_\Delta \frac{e^{-isw}}{2\pi}\, dw\,. \tag{59}$$

Hence

$$\left|\langle Z(\Delta)[b]\varphi, \psi\rangle_{N;2}\right| \leqslant \int_{\mathbb{R}} |g(t)|\, |J(t)|\, dt$$

where

$$J(t) = \int_{\mathbb{R}} \bar{g}(s+t)\, \langle e^{isa}\, b_t\varphi, \psi\rangle_{N;2}\, \delta_\Delta(s)\, ds \tag{60}$$

Now we observe that

$$J(t) = \delta_\Delta(h).$$

Using equations (31), (50), (51.a) and (52) yields

$$|J(t)| \leqslant C_{N,M} \|\psi\|\, \|\varphi\|_{\tilde{P};2}\, (1+|t|)^{2N+2M+P}\, \|g\|_{N+M;\infty}\,,$$

but then by using lemma III.15 we obtain

$$\left|\langle Z(\Delta)[b]\varphi,\psi\rangle_{N;2}\right| \leqslant C_{N,M} \|\psi\|\, \|\varphi\|_{\tilde{P};2} \|g\|_{N+M;\infty} \cdot (\pi 2^{2N+2M+P+2} \|g\|_{2N+2M+P+2;\infty})$$
$$\leqslant \tilde{C}_{N,M} \|\psi\| \|\varphi\|_{\tilde{P};2} \|g\|^2_{2N+2M+P+2;\infty}.$$

Thus

$$\|Z(\Delta)[b]\varphi\|_{N;2} \leqslant \tilde{C}_{N,M} \|\varphi\|_{\tilde{P};2}\, \|g\|^2_{2N+2M+P+2;\infty}\,. \tag{61}$$

This shows that for every Borel set $\Delta \in Bor(R)$, $Z(\Delta)$ is a linear map from $\mathcal{A}_h$ to itself.

We now show positivity. Let $b \in \mathcal{A}_+$, $\Delta \in Bor(R)$ and $\varphi \in S$ then as

$$\langle f_s(a)^* b f_s(a)\varphi, \varphi\rangle = \langle b f_s(a)\varphi, f_s(a)\varphi\rangle \geqslant 0$$

we have

$$+\infty > \langle Z(\Delta)[b]\varphi,\varphi\rangle = \int_\Delta \langle f_s(a)^* b f_s(a)\varphi,\varphi\rangle \, ds \geq 0 .$$

Hence, $Z(\Delta)[b] \in \mathcal{A}_+$.

The normalization condition is

$$Z(R)[I] = \int_{\mathbb{R}} f_s(a)^* f_s(a) \, ds$$
$$=1.$$

As $f \in S$ for each $s \in \mathbb{R}$ and $f_s(a)$ is bounded, the spectral calculus for bounded operators applies, changes of order of integration are allowed and as $\|f_s\|_2^2 = \int_{\mathbb{R}} |f_s(x)|^2 \, ds = 1$, the condition is verified.

Now we shall prove the requisite σ-additivity, eqn II.14 Let $\Phi \in \mathcal{A}_+'$, $b \in \mathcal{A}_+$ be arbitrary and $\{\Delta_j : j \geq 1\}$ be a family of mutually disjoint Borel sets. We use the spectral representation of $\hat{\Phi}$ and proposition II.30 to get

$$\Phi(Z(\Delta)[b]) = \sum_{n \geq 1} t_n \langle Z(\Delta)[b]e_n, e_n\rangle$$

for all Borel sets Δ .

The specific form of Z yields

$$\langle Z(\Delta)[b]e_n, e_n\rangle = \int_\Delta \langle f_w(a)^* b f_w(a) e_n, e_n\rangle \, ds$$

The integral is σ-additive, cf [39: exercise (29.6)], and so, changing back to the Z form,

$$\langle Z(\cup_j \Delta_j)[b]e_n, e_n\rangle = \sum_j \langle Z(\Delta_j)[b]e_n, e_n\rangle$$

for each n. By [25], theorem (8.3) we can interchange summation order, to get

$$\Phi(Z(\cup_j \Delta_j)[b]) = \sum_j \sum_n t_n \langle Z(\Delta_j)[b]e_n, e_n\rangle$$
$$= \sum_j \Phi(Z(\Delta_j)[b]).$$

By linearity this extends to all $\Phi \in \mathcal{A}_h'$, $b \in \mathcal{A}_h$, and so Z is σ-additive in the σ-topology.

Finally we shall show that $\int_{\mathbb{R}} t\ Z(dt)[I]$ is an element of $\mathcal{A}$, whence Z will have been proved to be an expectation. By the spectral calculus, for all $\varphi,\psi \in S$

$$\begin{aligned}
\langle Z(\Delta)[I]\varphi,\psi\rangle &= \int_{\Delta} \langle f_s(a)^* f_s(a)\varphi,\psi\rangle\, ds \\
&= \int_{\Delta}\int_{\mathbb{R}} |f_s(x)|^2\ \langle E(dx)\varphi,\psi\rangle\, ds \\
&= \int_{\mathbb{R}}\int_{\Delta} |f(x-s)|^2 ds\ \ \langle E(dx)\varphi,\psi\rangle \\
&= \int_{\mathbb{R}} F_{\Delta}(x)\ \langle E(dx)\varphi,\psi\rangle \\
&= \langle F_{\Delta}(a)\varphi,\psi\rangle \quad , \quad \Delta\in Bor(R) \qquad (62)
\end{aligned}$$

where

$$F_{\Delta}(x) = \int_{\Delta} |f(x-s)|^2\ ds.$$

As $0 \leqslant F_{\Delta}(t) \leqslant F_{\mathbb{R}}(t)$ and $F_{\mathbb{R}}(t)=1$, it follows that $0 \leqslant F_{\Delta}(a) \leqslant I$.

We can continuously extend $F_{\Delta}(a)$ from S to all of $L^2(\mathbb{R})$, whereby it is straightforward to see that, with

$$\langle F_{\Delta}(a)\varphi,\psi\rangle = \int_{\Delta} \langle f_s(a)^* f_s(a)\varphi,\psi\rangle\ ds \quad ;(\varphi,\psi\in L^2(\mathbb{R}))$$

$\Delta \longmapsto F_{\Delta}(a)$ is a generalized spectral family. This family defines a symmetric operator, call it

$$\begin{aligned}
X &= \int_{\mathbb{R}} t\ F_{dt}(a) \\
&\supset \int_{\mathbb{R}} t\ Z(dt)[I]
\end{aligned}$$

Now we must show that X belongs to $\mathcal{A}$. Firstly,

$$S \subset D(X) = \{\varphi\in L^2(R) : \int_{\mathbb{R}} t^2\ \langle F_{dt}(a)\varphi,\varphi\rangle < \infty\}.$$

But if $\varphi\in S$ then

$$\begin{aligned}
\int_{\mathbb{R}} t^2 \langle F_{dt}(a)\varphi,\varphi\rangle &= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} t^2\ |f(x-t)|^2 dt\Big)\langle E(dx)\varphi,\varphi\rangle \\
&= \Big[\int_{\mathbb{R}} t^2\ |f(t)|^2\ dt\Big]\langle\varphi,\varphi\rangle - 2\Big[\int_{\mathbb{R}} t\,|f(t)|^2\ dt\Big]\langle a\varphi,\varphi\rangle + \\
&\quad + \Big[\int_{\mathbb{R}} |f(t)|^2\ dt\Big]\langle a^2\varphi,\varphi\rangle.
\end{aligned}$$

Using lemma III.15 and the fact that $2|\langle a\varphi,\varphi\rangle| \leqslant \langle (I+a^2)\varphi,\varphi\rangle$

we obtain

$$\left|\int_{\mathbb{R}} t^2 \langle F_{dt}(a)\varphi\,,\varphi\,\rangle\right| \leqslant 2\,\|f\|^2_{2;2}\langle (I+a^2)\varphi\,,\varphi\,\rangle$$

which is finite.

Secondly we show that for all $\varphi\in S$, $X\varphi\in S$.

To see this, let $\varphi,\psi\in S$ be arbitrary. Then

$$\langle X\varphi\,,\psi\,\rangle = \int_{\mathbb{R}} t\langle F_{dt}(a)\,\varphi\,,\psi\rangle = \int_{\mathbb{R}}\int_{\mathbb{R}} t\,|f(x-t)|^2\langle E(dx)\varphi\,,\psi\,\rangle dt$$
$$= \langle a\,\varphi\,,\psi\,\rangle - [\int_{\mathbb{R}} t\,|f(t)|^2\,dt]\langle\,\varphi\,,\psi\,\rangle.$$

But then, by considering $\langle X\varphi\,,P^n\,Q^m\psi\rangle$ if $a=Q$, P or $\langle X\,\varphi\,,H^n\,Q^m\psi\rangle$ if $a=H$, and taking the supremum over ψ, $\|\psi\|=1$, we find that

$$\|X\varphi\|_{\widetilde{m,n};2} \leqslant \|a\varphi\|_{\widetilde{m,n};2} + \|f\|^2_{2;2}\,\|\varphi\|_{\widetilde{m,n};2}$$

and so $X\in\mathcal{A}_h$, proving the proposition. $\blacksquare$

III.16 <u>COROLLARY:</u> The $\langle\mathcal{A}\,,S\rangle$-measure determined by $Z[f,a,.]$ is

$$\Delta \longmapsto [\,\chi_\Delta *\,|f|^2\,](a) \qquad (63)$$

It is a covariant measurement in the sense of definition II.31. As it is bounded for all Δ, it is an element of $\mathcal{F}(E)$, cf. eqn II.(28.a), and so $\mathcal{S}[f;a;.\,]$ is an instrument for measuring a.

<u>PROOF.</u>

The first assertion follows from the fact that

$$\langle F_\Delta(a)\varphi\,,\psi\,\rangle = \int_{\mathbb{R}}(\int_\Delta |f(x-s)|^2 ds)\langle E(dx)\,\varphi\,,\psi\,\rangle$$
$$=\langle\,(\chi_\Delta *\,|f|^2\,)(a)\,\varphi\,,\psi\,\rangle.$$

The covariant property is shown as in [2], theorem 4.

The last assertion of the corollary follows from the boundedness of $F_\Delta(a)$ for all $\Delta\in \mathrm{Bor}(\mathbb{R})$, and the fact that

$$\Phi(F_\Delta(a))=\int_{\mathbb{R}}(\chi_\Delta*|f|^2)(t)\,\Phi(E(dt)) \qquad (64)$$

for each $\Phi\in\mathcal{A}_h'$. ∎

It is interesting to enquiry under what conditions the measures $\Delta\longmapsto E(\Delta)$ and $\Delta\longmapsto(\chi_\Delta*|f|^2)(a)$ give the same information. The answer is given in the next proposition and was shown by Davies [2].

III.17 <u>PROPOSITION:</u> If the Fourier transform of $|f|^2$ is non-zero on an open dense set then $E(.)$ and $(\chi_{(.)}*|f|^2)(a)$ give the same information.

This holds in particular if f is of compact support or if $|f|^2$ is of the form

$$|f|^2(x)=\text{const}\ \exp(-t\,x^2)$$

where $t>0$. ///

COMPOSITION OF INSTRUMENTS

As we mentioned previously, because $\mathcal{A}[u]$ is incomplete, a pre-instrument may, in general, extend to a post-instrument.
This applies in particular to the compose of two instruments. We now have a class of instruments $\mathcal{J}[f;a;\Delta]$ for $f\in S$, $a=P,Q,H$. By computing the compositions explicitly we shall show that the compose of any two, hence any finite number, of these instruments is an instrument.
As regards the implications for the general case, we believe that this result is special, and depends on the translation covariance of these instruments and various special properties.

III.18 **PROPOSITION:** Let $Z_j[f_j;a_j:\Delta]$ $(j=1,2)$ be expectations of the type described in Proposition III.1. Then $Z_{21}=Z_2\circ Z_1$ is an expectation.

PROOF.

Let us specialize to $a_1=Q$ and $a_2=H$ for definiteness. The other cases are proved similarly.

By using twice equation (58), first for $a_1=Q$ and then for $a_2=H$ we get

$$\langle Z_{21}(\Delta_2\times\Delta_1)[b]\varphi,\psi\rangle=\int_{\mathbb{R}^2} g_1(t)g_2(\xi)\int_{\mathbb{R}^2}\bar{g}_1(s+t)\,\bar{g}_2(\eta+\xi)\,\langle e^{isH}e^{itH}(e^{i\eta Q}b_\xi)e^{-itH}\varphi,\psi\rangle\,.$$
$$\cdot\,\delta_{\Delta_2\times\Delta_1}(\eta,s)\,ds\,d\eta\,d\xi\,dt\,. \tag{65}$$

where
$$b_\xi=e^{i\xi Q}\,b\,e^{-i\xi Q} \tag{66}$$

and we have used the fact that

$$\delta_{\Delta_2\times\Delta_1}(\eta,s) = \delta_{\Delta_2}(\eta)\ \delta_{\Delta_1}(s) \tag{67}$$

Hence (65) is satisfied for every Borel rectangle in $\mathbb{R}^2$.

Now consider the set function $\tilde{Z}(\cdot)$, defined on all Borel subsets Δ of $\mathbb{R}^2$ by

$$\langle\tilde{Z}(\Delta)[b]\varphi,\psi\rangle = \int_{\mathbb{R}^2} g_1(t)g_2(\xi)\int_{\mathbb{R}^2}\bar{g}_1(s+t)\bar{g}_2(\eta+\xi)\langle e^{isH}e^{itH}(e^{-i\eta Q}b_\xi)e^{-itH}\varphi,\psi\rangle\cdot \delta_\Delta(\eta,s)\,ds\,d\eta\,d\xi\,dt \tag{68}$$

where $\varphi,\psi\in S$, $b\in \mathcal{A}_h$.

Clearly $\tilde{Z}(\Delta)[\cdot]$ is linear for each $\Delta\in \mathrm{Bor}(\mathbb{R}^2)$. Moreover

$$|\langle\tilde{Z}(\Delta)[b]\varphi,\psi\rangle_{N;2}| \le \int_{\mathbb{R}^2}|g_1(t)g_2(\xi)|\,|J(t,\xi)|\,d\xi\,dt \tag{69}$$

where

$$J(t,\xi) = \int_{\mathbb{R}^2}\bar{g}_1(s+t)\bar{g}_2(\eta+\xi)\langle e^{isH}e^{itH}(e^{-i\eta Q}b_\xi)e^{-itH}\varphi,\psi\rangle\,\delta_\Delta(\eta,s)\,ds\,d\eta \tag{70}$$

Now we observe that

$$J(t,\xi) = \delta_\Delta(h)$$

where h is as in Lemma III.20.

Using the Lemma just quoted and Lemma III.19 we obtain

$$|J(t,\xi)| \le \tilde{c}_{N,M}\|\psi\|\,\|\varphi\|_{I;2}(1+|t|)^{2N+3M+T}(1+|\xi|)^{2(P+M)+R}\cdot \|g_1\|_{N+2M;\infty}\,\|g_2\|_{P+M;\infty} \tag{71}$$

where P, R and T depend on M and N only.

But then by using Lemma III.15 we obtain

$$|\langle\tilde{Z}(\Delta)[b]\varphi,\psi\rangle_{N;2}| \le \tilde{c}_{N,M}\|\psi\|\,\|\varphi\|_{I;2}\ \|g_1\|_{N+2M;\infty}\,\|g_2\|_{P+M;\infty}\cdot (\pi\, 2^{2N+3M+T}\|g_1\|_{2N+3M+T+2;\infty})\cdot(\pi\, 2^{2(P+M)+R}\|g_2\|_{2(P+M)+R+2;\infty}) \tag{72}$$

Thus

$$\|\tilde{Z}(\Delta)[b]\varphi\|_{N;2} \le d_{N,M}\,\|\varphi\|_{I;2}\,\|g_1\|^2_{2N+3M+T+2;\infty}\,\|g_2\|^2_{2(P+M)+R+2;\infty} \tag{73}$$

This shows that for every $\Delta \in Bor(\mathbb{R}^2)$, $\tilde{Z}(\Delta)$ is a linear map from $\mathcal{A}_h$ to itself.

Now $\tilde{Z}(\cdot)$ coincides with $Z_{21}(\cdot)$ on the Borel rectangles of $\mathbb{R}^2$, hence $\tilde{Z}(\cdot)$ is an extension of $Z_{21}(\cdot)$ defined on all Borel subsets Δ of $\mathbb{R}^2$. By Proposition II.16 the extension of $Z_{21}(\cdot)$ to $Bor(\mathbb{R}^2)$ is unique, therefore $\tilde{Z}(\Delta) = Z_{21}(\Delta)$ for all Borel subsets Δ of $\mathbb{R}^2$. This shows that Z_{21} is an expectation. ∎

In the proof of the above proposition we have used the following generalized version of Lemma III.11 and Corollary III.13.

In the sequel we use the following set of seminorms for $S(\mathbb{R}^2)$, c.f. [26].

$$\|f\|_{M;\infty} = \max\{\|f\|_{m,n;\infty} : |m| \leq M, |n| \leq M-|m|\} \tag{74}$$

$$\|f\|_{M;2} = \max\{\|f\|_{m,n;2} : |m| \leq M, |n| \leq M-|m|\} \tag{75}$$

where m,n are 2-index, i.e. $m=(m_1,m_2)$, $m_1,m_2 \in \mathbb{N}_0$.

III.19 <u>LEMMA</u>: For any set $\Delta \in Bor(\mathbb{R}^2)$, the Fourier transform of χ_Δ, $2\pi\delta_\Delta$, is a tempered distribution. Hence, there exists an index M and a constant C (both independent of Δ), such that for every $f \in S(\mathbb{R}^2)$

$$|\delta_\Delta(f)| \leq C\|f\|_{M;\infty} \tag{76}$$

c.f. Lemma III.11. ///

III.20 <u>LEMMA</u>: Let $\varphi, \psi \in S(\mathbb{R})$, $b \in \mathcal{A}_h$, $g_k \in S(\mathbb{R})$ $(k=1,2)$. Set

$$b_\xi = e^{i\xi Q} b e^{-i\xi Q} \quad , \xi \in \mathbb{R}. \tag{77}$$

The function , $(N\in\mathbb{N}_0)$

$$(s,\eta)\mapsto h(s,\eta)=\bar{g}_1(s+t)\,\bar{g}_2(\eta+\xi)\,\langle e^{isH}e^{itH}(e^{i\eta Q}b_\xi)e^{-itH}\varphi,\psi\rangle_{N;2} \qquad (78)$$

belongs to $S(\mathbb{R}^2)$, and

$$\|h\|_{M;\infty}\leq c_{N,M}\|\psi\|\|\varphi\|_{\underline{T};2}(1+|t|)^{2N+3M+T}.(1+|\xi|)^{2(P+M)+R}\|g_1\|_{N+2M;\infty}\|g_2\|_{P+M;\infty} \qquad (79)$$

where P, R and T depend on N and M only.

<u>PROOF.</u>

We must show that for any $M\in\mathbb{N}_0$

$$\|h\|_{M;\infty}=\max\{\sup_{s,\eta\in\mathbb{R}}|s^m\eta^n D_s^q D_\eta^p h(s,\eta)| : m+n\leq M,\ p+q\leq M-(m+n)\}<\infty \qquad (79)$$

Let us abbreviate to $\langle\ ,\ \rangle_{N;2}$ the inner product appearing as a factor in (78). Then

$$D_\eta^p\langle\ ,\ \rangle_{N;2}=(i)^p\langle e^{isH}e^{itH}(e^{i\eta Q}Q^p b_\xi)e^{-itH}\varphi,\psi\rangle_{N;2} \qquad (80)$$

$$D_s^q D_\eta^p\langle\ ,\ \rangle_{N;2}=i^{p+q}\langle H^q e^{isH}e^{itH}(e^{i\eta Q}Q^p b_\xi)e^{-itH}\varphi,\psi\rangle_{N;2}\ , \qquad (81)$$

thus

$$|D_s^q D_\eta^p\langle\ ,\ \rangle_{N;2}|\leq\|\psi\|\,\|H^q e^{isH}e^{itH}(e^{i\eta Q}Q^p b_\xi)e^{-itH}\varphi\|_{\underline{N};2}$$
$$=\|\psi\|\,\|e^{isH}e^{itH}(e^{i\eta Q}Q^p b_\xi)e^{-itH}\varphi\|_{\underline{N+q};2} \qquad (82)$$

Now

$$\|h\|_{M;\infty}\leq\max\{\sum_{r_1=r_2=0}^{q,p}\binom{q}{r_1}\binom{p}{r_2}\sup_{s,\eta\in\mathbb{R}}|s^m\eta^n D_s^{r_1}\bar{g}_1(s+t)D_\eta^{r_2}\bar{g}_2(\eta+\xi)D_s^{q-r_1}D_\eta^{p-r_2}\langle\ ,\ \rangle_{N;2}| :$$
$$m+n\leq M,\ p+q\leq M-(m+n)\} \qquad (83)$$

Using (82) we find a bound for the term $\sup_{s,\eta\in\mathbb{R}}|*|$ of the above expression

$$\sup_{s,\eta\in\mathbb{R}}|*|\leq\|\psi\|\sup_{s,\eta\in\mathbb{R}}|s^m\eta^n D_s^{r_1}\bar{g}_1(s+t)D_\eta^{r_2}\bar{g}_2(\eta+\xi)|\,\|e^{isH}e^{itH}(e^{i\eta Q}Q^p b_\xi)e^{-itH}\varphi\|_{\underline{N+q};2}$$
$$\leq\|\psi\|\sup_{s,\eta\in\mathbb{R}}|s^m\eta^n D_s^{r_1}\bar{g}_1(s+t)D_\eta^{r_2}\bar{g}_2(\eta+\xi)|\,\|e^{isH}e^{itH}(e^{i\eta Q}Q^p b_\xi)e^{-itH}\varphi\|_{\underline{N+M};2}$$

Using Lemmas III.8 , III.14 ; Corollaries III.10, III.13, and the equivalence of the system of norms $\|.\|_{\underline{n};2}$ and $\|.\|_{n;2}$ we obtain

$$\|e^{isH}e^{itH}(e^{i\eta Q}Q^p b_\xi)e^{-itH}\varphi\|_{\underline{N+M};2} \leq$$
$$\leq c_{N,M}[(1+|s|)(1+|t|)]^{N+M}\|e^{i\eta Q}Q^p b_\xi e^{-itH}\varphi\|_{\underline{N+M};2}$$
$$\leq d_{N,M}[(1+|s|)(1+|t|)]^{N+M}\|e^{i\eta Q}Q^p b_\xi e^{-itH}\varphi\|_{P;2}$$

(P depends on N and M only)

$$\leq d_{N,M}[(1+|s|)(1+|t|)]^{N+M}(1+|\eta|)^P\|Q^p b_\xi e^{-itH}\varphi\|_{P;2}$$
$$\leq d_{N,M}[(1+|s|)(1+|t|)]^{N+M}(1+|\eta|)^P\|b_\xi e^{-itH}\varphi\|_{P+p;2}$$
$$\leq d_{N,M}[(1+|s|)(1+|t|)]^{N+M}(1+|\eta|)^P\|b_\xi e^{-itH}\varphi\|_{P+M;2}$$
$$\leq \tilde{d}_{N,M}[(1+|s|)(1+|t|)]^{N+M}(1+|\eta|)^P(1+|\xi|)^{P+M+R}\|e^{-itH}\varphi\|_{R;2}$$

(R depends on P and M only)

$$\leq \tilde{d}_{N,M}[(1+|s|)(1+|t|)]^{N+M}(1+|\eta|)^P(1+|\xi|)^{P+M+R}\|e^{-itH}\varphi\|_{\underline{T};2}$$

(T depends on R only)

$$\leq \hat{d}_{N,M}[(1+|s|)(1+|t|)]^{N+M}(1+|\eta|)^P(1+|\xi|)^{P+M+R}(1+|t|)^T\|\varphi\|_{\underline{T};2}$$
$$= e_{N,M}(1+|s|)^{N+M}(1+|t|)^{N+M+T}(1+|\eta|)^P(1+|\xi|)^{P+M+R}\|\varphi\|_{\underline{T};2}$$

Thus

$$\sup_{s,\eta\in\mathbb{R}}|*| \leq \|\psi\| e_{N,M}(1+|t|)^{N+M+T}(1+|\xi|)^{P+M+R}\|\varphi\|_{\underline{T};2}\cdot\sup_{s,\eta\in\mathbb{R}}|s^m\eta^n(1+|s|)^{N+M}\cdot$$
$$(1+|\eta|)^P D_s^{r_1}\bar{g}_1(s+t)D_\eta^{r_2}\bar{g}_2(\eta+\xi)|$$
$$\leq \|\psi\|\|\varphi\|_{\underline{T};2}e_{N,M}(1+|t|)^{N+M+T}(1+|\xi|)^{P+M+R}\cdot$$
$$\cdot\underbrace{\sup_{s\in\mathbb{R}}|s^m(1+|s|)^{N+M}D_s^{r_1}\bar{g}_1(s+t)|}_{\#1}\cdot\underbrace{\sup_{\eta\in\mathbb{R}}|\eta^n(1+|\eta|)^P D_\eta^{r_2}\bar{g}_2(\eta+\xi)|}_{\#2}$$

But (see eqns (43),(44))

$$\#1 \leq 2^{N+M}\|\bar{g}_{1,-t}\|_{N+M+m+r_1;\infty}$$
$$\leq 2^{N+M}\|\bar{g}_{1,-t}\|_{N+2M;\infty}$$
$$\leq 2^{N+M}(1+|t|)^{N+2M}\|g_1\|_{N+2M;\infty}$$

Similarly

$$\#2 \leq 2^P(1+|\xi|)^{P+M}\|g_2\|_{P+M;\infty}$$

Hence

$$\sup_{s,\eta\in\mathbb{R}} | * | \le \|\psi\|\|\varphi\|_{I;2}\,\tilde{e}_{N,M}(1+|t|)^{2N+3M+T}(1+|\xi|)^{2(P+M)+R}\cdot$$
$$\cdot\,\|g_1\|_{N+2M;\infty}\ \|g_2\|_{P+M;\infty}$$

Finally

$$\|h\|_{M;\infty} \le \tilde{e}_{N,M}\|\psi\|\|\varphi\|_{I;2}(1+|t|)^{2N+3M+T}(1+|\xi|)^{2(P+M)+R}\cdot$$
$$\cdot\,\|g_1\|_{N+2M;\infty}\ \|g_2\|_{P+M;\infty}\,.\max\Big\{\sum_{r_1=r_2=0}^{q,p}\binom{q}{r_1}\binom{p}{r_2} : p+q\le M\Big\}$$

Using the fact that

$$\sum_{r_1=r_2=0}^{q,p}\binom{q}{r_1}\binom{p}{r_2} = 2^q 2^p \le 2^M ,$$

yields the desired result. $\blacksquare$

SUMMARY

In this chapter we construct a special class of instruments for measuring Q, P, and H on S(R). These instruments are just those of Davies' approximate position observables,but using suitable smoothy functions.

The technical problem resides in proving that the Davies formula does indeed define an instrument in our sense.

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CONCLUSIONS

We have based our theory on the choice of a nuclear space generated by a suitable self-adjoint operator,for the space W of pure states and $\mathcal{A} = \mathcal{L}^{+}(W)$ for the algebra of observables . In consequence, all states are normal, and positivity implies continuity.

Expectations are maps Z: Bor(R) $\longrightarrow L_{+}(\mathcal{A})$; and instruments are maps $\mathcal{S}$: Bor(R) $\longrightarrow \mathcal{L}_{+}(\mathcal{A}_{h}^{\prime}[\sigma^{*}])$ such that there is an expectation Z with $\mathcal{S} = Z^{t}$ .Each instrument defines a unique $(\mathcal{A},W)$-measure M , which in turn defines an observable b. Introducing the partial order 'less information than ' on the set of observables , each such instrument measures all observables for which $b \prec a$.

We show that instruments exist to measure all observables a for which $b \prec a$ for some b as above. We also show that reasonable instruments exist to measure Q, P, and H on S(R):these are based on Davies' approximate position measurements.We leave open the question of a measure of fidelity for instruments. we also show that instruments compose to pre-instruments,which in turn extend to post-instruments: thus composition is not a relation on the set of instruments.We leave open the question of various degrees of repeatability,save to note that strong repeatability, ie, in the Von Neumann sense,is not generally possible.

In conclusion we have shown that it is possible to base a theory of quantum measurements on the Op*-algebra framework, but at the expense of increasing the technical complications. Various interesting questions remain open, but it has been shown that a suitable generalization of Von Neumann's theory is compatible with choosing $\mathcal{L}^{+}(\mathcal{D})$ as the algebra of observables.

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